

# On Integrable Systems and Supersymmetric Gauge Theories

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## Abstract

The properties of the  $\mathcal{N} = 2$  SUSY gauge theories underlying the Seiberg-Witten hypothesis are discussed. The main ingredients of the formulation of the finite-gap solutions to integrable equations in terms of complex curves and generating 1-differential are presented, the invariant sense of these definitions is illustrated. Recently found exact nonperturbative solutions to  $\mathcal{N} = 2$  SUSY gauge theories are formulated using the methods of the theory of integrable systems and where possible the parallels between standard quantum field theory results and solutions to integrable systems are discussed.

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# 1 Introduction: Main Definitions

The aim of this paper is to present in a clear form the main ideas of the relations between the exact solutions to the  $\mathcal{N} = 2$  supersymmetric (SUSY) Yang-Mills theory (arising in the point-particle limit of string theory) and integrable systems. The approach to the Seiberg-Witten effective theory based on integrable systems was proposed in [1] and developed along these lines in [2]-[19], where all necessary details can be found.

The plan of the paper looks as follows. First, I review what is known about construction of the effective actions for the low-energy  $\mathcal{N} = 2$  Yang-Mills theories and consider the definition of the algebro-geometric solution to an integrable system in terms of a complex curve and generating 1-differential. These definitions are illustrated the basic example of the Toda chain periodic solutions which is considered in detail. Next, I pass to the Seiberg-Witten solutions directly and show that they are indeed defined by the same data as the finite-gap solutions to integrable systems, though the complete formulation requires to introduce *deformations* of the finite-gap solutions. Finally, the explicit differential equations and direct computations of the prepotential of the effective theory are presented and compared when possible with the well-known computations from supersymmetric quantum gauge theories.

## 1.1 Seiberg-Witten effective theory

Let us start with some introductory motivations and present the main definitions which will be used below for the formulation of the effective Seiberg-Witten theory in terms of integrable systems.

The object of study is given by the effective (abelian)  $\mathcal{N} = 2$  supersymmetric gauge theories in four (or five-) <sup>1</sup> dimensions corresponding to the  $\mathcal{N} = 2$  SUSY Yang-Mills theory with the bare action

$$\mathcal{L} = \int d^4\theta \hat{\mathcal{L}}(\Phi) = \dots \frac{1}{g^2} \text{Tr} \mathbf{F}_{\mu\nu}^2 + i\theta \text{Tr} \mathbf{F}_{\mu\nu} \tilde{\mathbf{F}}_{\mu\nu} + \dots \quad (1)$$

The exact nonperturbative results [20, 21, 23, 24] contain information about the spectrum of massive BPS excitations ("W-bosons" and monopoles) <sup>2</sup> and the *Wilsonian* effective action for the massless particles (see for example [25]). The most essential feature of this formulation is that the effective action can be written in terms of a single (holomorphic) *function* of several complex variables [20, 21]. Later on, according to the standard terminology, this function will be referred to as *prepotential*.

For the  $\mathcal{N} = 2$  SUSY gauge theory the result can be understood in the following way. The scalar potential in the  $\mathcal{N} = 2$  SUSY gauge theory action has the form  $V(\phi) = \text{Tr}[\phi, \phi^\dagger]^2$  and its minima after factorization over the gauge group correspond to the diagonal  $([\phi, \phi^\dagger] = 0)$ , in the theory with  $SU(N_c)$  gauge group traceless

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<sup>1</sup>Let us notice immediately that the effective formulation of the exact nonperturbative solutions we discuss here does not depend explicitly on the dimension of target-space. Integrable systems suggest a *universal* effective formulation which is known at present for some 2D string models as well as for the theories considered in this paper.

<sup>2</sup>The BPS ( $\equiv$  the Bogomolny-Prasad-Sommerfeld) states are the states in the "small" multiplet with masses being proportional to the central charges of the extended  $\mathcal{N} \geq 2$  algebra of supersymmetry.

matrices

$$\phi = \left( \begin{array}{cccc} A_1 & & & \\ & A_2 & & \\ & & \dots & \\ & & & A_{N_c} \end{array} \right) \Bigg|_{\text{Tr} \phi = \sum A_j = 0} \quad (2)$$

whose invariants

$$\det(\lambda - \phi) = P_{N_c}(\lambda) = \sum_{k=0}^{N_c} s_{N_c-k} \lambda^k \quad (3)$$

(the total number of algebraically independent ones is  $\text{rank } SU(N_c) = N_c - 1$ ; one can also take any other set of invariants,) parameterize the moduli space of the theory. Due to the Higgs effect the off-diagonal part of the gauge field  $\mathbf{A}_\mu$  becomes massive, since

$$[\phi, \mathbf{A}_\mu]_{ij} = (A_i - A_j) \mathbf{A}_\mu^{ij} \quad (4)$$

while the diagonal part, as it follows from (103) remains massless, i.e. the gauge group  $G = SU(N_c)$  breaks down to  $U(1)^{\text{rank } G} = U(1)^{N_c-1}$ . Thus, the effective  $\mathcal{N} = 2$  abelian gauge theory arises with the effective Lagrangian, written in terms of the superfields

$$\begin{aligned} \Phi_i &= \varphi^i + \vartheta \sigma_{\mu\nu} \tilde{\vartheta} f_{\mu\nu}^i + \dots \\ \sigma_{\mu\nu} &\sim [\gamma_\mu, \gamma_\nu] \end{aligned} \quad (5)$$

whose vacuum values coincide with the diagonal values of (2). Therefore the function of complex variables  $\mathcal{F}(a) = \mathcal{F}(A)|_{\sum A_i=0}$ , indeed determines the Wilsonian effective action for the massless particles by means of the following substitution

$$\mathcal{L}_{\text{eff}} \sim \text{Im} \int d^4 \vartheta \mathcal{F}(A_i \rightarrow \Phi_i) = \dots \text{Im} \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j}(a) f_{\mu\nu}^i f_{\mu\nu}^j + \dots \quad (6)$$

This fact can be checked by explicit computations of quantum corrections in conventional  $\mathcal{N} = 2$  SUSY gauge theory.

As for the massive excitations, it turns out [20, 21], that at least the BPS massive spectrum, i.e. the spectrum of states of "small" multiplet whose masses are proportional to the central charges of the extended  $\mathcal{N} \geq 2$  SUSY algebra, is related with the prepotential  $\mathcal{F}$  by  $M \sim |\mathbf{n}\mathbf{a} + \mathbf{m}\mathbf{a}_D|$ , where  $\mathbf{a}_D = \frac{\partial \mathcal{F}}{\partial \mathbf{a}}$ . According to the Seiberg-Witten hypothesis (which will be formulated strictly below in subsect.1.3) the BPS masses  $\mathbf{a}$  and  $\mathbf{a}_D$  can be expressed through the periods of a meromorphic differential on auxiliary Riemann surface and depend on the vacuum expectation values of scalar fields, which can be considered as certain co-ordinates on the moduli space of auxiliary surface. For example, in the case of pure SUSY gauge theory with the  $SU(N_c)$  gauge group the auxiliary curve and meromorphic differential have the form [20, 23]

$$w + \frac{1}{w} = 2P_{N_c}(\lambda) \quad dS = \lambda \frac{dw}{w} \quad (7)$$

while for the  $\mathcal{N} = 2$  SUSY QCD [21, 24]

$$W + \frac{1}{W} = \frac{2P_{N_c}(\lambda)}{P_{N_f}(\lambda)} \quad dS = \lambda \frac{dW}{W} \quad (8)$$

Thus the knowledge of the function  $\mathcal{F}$  and its derivatives as functions on moduli (and also as functions of possible external sources  $\mathbf{T}$ ) gives the most complete up to now information about the theory. It will be demonstrated below that the fact that the nonperturbative solution to the  $\mathcal{N} = 2$  SUSY gauge theory can be presented in terms of effective integrable system gives rise to the main property of the  $\mathcal{F}$ -function, that it depends on (part of) its variables as on (some) moduli of the complex structures of (auxiliary) complex curves or Riemann surfaces (7), (8).

## 1.2 Integrable systems

The main idea is to identify function  $\mathcal{F}$  and other characteristics of effective theory with the objects from the theory of integrable systems. Fortunately, it turns out that this particular class of effective theories (as well as the class of low-dimensional string models) can be described in terms of the integrable systems of (Kadomtsev-Petviashvili) KP and Toda type. The starting point is that the KP

$$\frac{\partial^2 U}{\partial T_2^2} = \frac{\partial}{\partial T_1} \left( \frac{\partial U}{\partial T_3} + U \frac{\partial U}{\partial T_1} + \frac{\partial^3 U}{\partial T_1^3} \right) \quad (9)$$

and the Toda lattice

$$\frac{\partial^2 \phi_n}{\partial T_1 \partial \overline{T}_1} = e^{\phi_{n+1} - \phi_n} - e^{\phi_n - \phi_{n-1}} \quad (10)$$

equations (and other equations of the same class: to be referred to later as KP/Toda type equations) possess the infinite amount of integrals of motion which can be considered as generators of the (mutually commuting and of course commuting with the first flows (9) and (10)) infinite amount (hierarchy) of flows parameterized by "elder" times. The differential equations describing  $T_k$ -flows have a complicated form if written for the functions (potentials)  $U(\mathbf{T})$  and  $\phi_n(\mathbf{T})$ , but there exists much more elegant way of presenting the whole picture.

This way is based on using the auxiliary linear problem for the hierarchy of nonlinear equations

$$\frac{\partial}{\partial T_k} \Psi = B_k \Psi \quad (11)$$

where  $B_k = B_k[U]$  (or  $B_k = B_k[\phi]$ ) are the differential operators *only* in  $T_1$  for the KP case (9) or difference operators (with respect to the discrete time  $n$ ) in the case of Toda lattice (10). The solution  $\Psi$  to the auxiliary linear problem is called usually the Baker-Akhiezer (BA) function if the equations (11) are supplemented by the Lax equation

$$\mathcal{L}\Psi = \lambda\Psi \quad (12)$$

which in the case of reductions of the KP/Toda hierarchies can be considered as one of the equations of the tower (11). In such language the hierarchy of nonlinear equations consisting of a "tower" built above (9), (10) becomes equivalent to a single (operator) Lax equation

$$\frac{\partial \mathcal{L}}{\partial T_k} = [B_k, \mathcal{L}] \quad (13)$$

or to the consistency conditions (the Zakharov-Shabat equations)

$$\left[ \frac{\partial}{\partial T_k} - B_k, \frac{\partial}{\partial T_l} - B_l \right] = 0 \quad (14)$$

The most universal object in such formulation is given by so called Hirota's  $\tau$ -function, satisfying the infinite chain of the bilinear equations (the Hirota equations) and generating solutions to the hierarchy, the BA functions

etc. For example, in the case of KP hierarchy the BA function  $\Psi(\lambda, \mathbf{T})$  and the "potential"  $U(\mathbf{T})$  are expressed through the  $\tau$ -function in the following way (in order to avoid misleading below  $\tau$ -functions will be denoted as  $\mathcal{T}$  while notation  $\tau$  is reserved for the modular parameter of elliptic curve):

$$\begin{aligned}\Psi &= e^{\sum T_k \lambda^k} \frac{\mathcal{T}(T_k - \frac{1}{k\lambda^k})}{\mathcal{T}(\mathbf{T})} \\ U(\mathbf{T}) &= \partial^2 \log \mathcal{T}(\mathbf{T}) \equiv \frac{\partial^2}{\partial T_1^2} \log \mathcal{T}(\mathbf{T})\end{aligned}\tag{15}$$

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and one can easily find analogous formulas for the Toda lattice (see for example [30]) and other integrable systems.

The KP/Toda systems have infinite amount of solutions parameterized by so called infinite-dimensional Grassmannian (roughly speaking a function of two variables – the initial conditions) [31, 32]. The particular solutions can be distinguished by additional (sometimes linear) equations, say, on the  $\tau$ -function.

A special role is played by *finite-dimensional* solutions to the hierarchies of integrable equations when only the finite number of the integrals of motion (and flows  $\frac{\partial}{\partial T_k}$ ) is algebraically independent. The important example of such finite-dimensional integrable systems is given by the finite-gap solutions, usually fixed by the Novikov constraint [27]

$$\begin{aligned}[\mathcal{L}, \mathcal{A}] &= 0 \\ \mathcal{A} &= \sum_k^{\text{finite}} c_k B_k\end{aligned}\tag{16}$$

where  $\mathcal{L}$  is the Lax operator (12),  $B_k$  are the evolution operators (11), and  $c_k$  – some *finite* set of nonzero constants. The integration of a generic finite-gap problem is given by the Krichever construction [26] and consists of the following steps<sup>3</sup>:

- The common spectrum of the commutative operators  $\mathcal{L}$  and  $\mathcal{A}$  (16) can be described by a system of equations, giving rise to a complex algebraic curve  $\Sigma$ .
- The BA function is a section of some bundle over  $\Sigma$  – in our case it will be almost always a *line* bundle.
- The moduli of a complex curve are the integrals of motion of the system (16).
- The integrable change of variables is given by the Abel map, and the Liouville torus (angle variables) is a real section of the Jacobian of  $\Sigma$ .
- The Hamiltonian structure of the finite-gap solution can be formulated with the help of generating (meromorphic on  $\Sigma$ ) 1-differential  $dS$ , whose periods (the integrals over nontrivial cycles on Riemann surface) are canonically normalized integrals of motion of given dynamical system.

The arising complex curves are usually described by the algebraic equations

$$\mathcal{P}(\lambda, w) = 0\tag{17}$$

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<sup>3</sup>Here only a "rough picture" of the Krichever construction is presented, the exact mathematical definitions and theorems can be found in [26, 27, 28, 29, 32].

(one relation on two complex variables (17), where  $\mathcal{P}$  – a polynomial, whose coefficients are moduli of the complex structure, defines a  $1_{\mathbf{C}d}$  complex or  $2_{\mathbf{R}d}$  real manifold) or by system of equations on several complex variables. Topologically any complex curve is characterized by a single non-negative integer parameter – genus  $g$  (the number of handles), and for a given genus the complex structure is parameterized by  $3g - 3$  complex numbers – the moduli of the complex structure  $\dim_{\mathbf{C}} \mathcal{M}_g = 3g - 3$ .

The finite gap integrable systems usually correspond to the  $g$ -parametric families of complex curves (so that the dimension of the corresponding submanifold in the moduli space is equal to the dimension of the Jacobian, i.e. the number of independent integrals of motion coincides with the number of angle variables <sup>4</sup>). The dimension of the Jacobian is determined by the total number of (globally defined) holomorphic differentials  $d\omega_i$ ,  $i = 1, \dots, g$ , and is equal to the genus of  $\Sigma_g$ . On  $\Sigma_g$  there exists  $2g$  independent noncontractable contours (two around each handle) which can be canonically split into so called  $A_i$ ,  $i = 1, \dots, g$ , and  $B_i$ ,  $i = 1, \dots, g$  cycles with the intersection index  $A_i \circ B_j = \delta_{ij}$ . The holomorphic differentials are usually taken to be normalized to the **A**-cycles

$$\oint_{A_j} d\omega_i = \delta_{ij} \quad (18)$$

then the integrals of the **B**-cycles give the period matrix

$$\begin{aligned} \oint_{B_j} d\omega_i &= T_{ij} \\ \int_{\Sigma_g} d\omega_i \wedge \overline{d\omega_j} &= \text{Im} T_{ij} \end{aligned} \quad (19)$$

As it is well known the period matrix (19) is symmetric which can be checked by application of the Stokes theorem to

$$0 = \int_{\Sigma_g} d\omega_i \wedge d\omega_j = \sum_{k=1}^g \oint_{A_k} d\omega_i \oint_{B_k} d\omega_j - (i \leftrightarrow j) = T_{ij} - T_{ji} \quad (20)$$

The derivatives of generating differential  $dS$  over  $g$  directions in moduli space, corresponding to the integrals of motion give rise to (some) holomorphic differentials

$$\frac{\partial dS}{\partial h_k} \sim dv_k \quad (21)$$

where in fact the canonical holomorphic differential appear only if one takes as co-ordinates on moduli space the canonically normalized integrals of motion – **A**-periods of the differential  $dS$

$$\mathbf{a} = \oint_{\mathbf{A}} dS \quad (22)$$

By accepted convention the corresponding "dual" **B**-periods will be called  $\mathbf{a}_D$

$$\mathbf{a}_D = \oint_{\mathbf{B}} dS \quad (23)$$

The existence of the relation (21) can be trivially checked for all known  $g$ -parametric families of curves. In fact it is related to the existence of specific co-ordinates on moduli space satisfying the consistency condition

$$\frac{\partial dv_k}{\partial h_l} = \frac{\partial dv_l}{\partial h_k} \quad (24)$$

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<sup>4</sup>In fact the  $g$ -parametric families arise in the simplest cases (in the context of this paper these are the theories with the  $SU(N_c)$  gauge group). In general situation one should consider the Prym manifold arising as a "factor" of Jacobian over some involution (see for explicit examples [17] and references therein).

This property will be discussed in more general context of the Whitham hierarchy. The finite-gap solutions are the most simple examples of solutions to integrable equations which are related directly to the nonperturbative quantum theories. In fact they should be considered only as some "approximations" to the exact solutions, which allow one however to describe part of the *exact* physical characteristics of the theory, mostly concerning its massless sector. Moreover, in many cases (e.g. in 2D string theories) the exact solutions can be considered as integrable deformations of the finite-gap solutions, described in terms of the Whitham hierarchies.

### 1.3 Seiberg-Witten map

Now we can pass to the exact formulation of the Seiberg-Witten effective theory which can be formally defined as a map

$$G, \tau, h_k \rightarrow a_i, a_i^D \quad (25)$$

( $G$  is gauge group,  $\tau$  – the UV coupling constant,  $h_k = \frac{1}{k} \langle \text{Tr} \Phi^k \rangle$  – the v.e.v.'s of the Higgs field) and has an elegant description in terms of an integrable system. In most known cases the integrable system is described in terms of a complex curve  $\Sigma_g$  with  $h_k$  parameterizing some of the (in most cases hyperelliptic) moduli of complex structures. The map (25) is described in terms of the periods (22), (23) of the meromorphic 1-form (21) which determine the BPS massive spectrum,

$$M \sim |\mathbf{n}\mathbf{a} + \mathbf{m}\mathbf{a}^D| \quad (26)$$

and the prepotential  $\mathcal{F}$  by the defining equation

$$a_i^D = \frac{\partial \mathcal{F}}{\partial a_i} \quad (27)$$

The derivatives give the period matrix having the sense of the low-energy coupling constants for the abelian gauge fields (cf. with (6)): indeed, from (22), (23) and (19) it follows that

$$\begin{aligned} \delta_{ij} &= \frac{\partial a_i}{\partial a_j} = \oint_{A_i} \frac{\partial dS}{\partial a_j} = \oint_{A_i} d\omega_j \\ \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} &= \frac{\partial a_i^D}{\partial a_j} = \oint_{B_i} \frac{\partial dS}{\partial a_j} = \oint_{B_i} d\omega_j = T_{ij} \end{aligned} \quad (28)$$

The curves  $\Sigma_{g=\text{rank } G}$  are special spectral curves of the nontrivial finite-gap solutions to the periodic Toda-chain problem and its natural deformations. The main object to be considered below – the prepotential

$$\mathcal{F} = \log \mathcal{T} \quad (29)$$

is logarithm of the  $\tau$ -function of the Whitham hierarchy, associated to a particular finite gap solution.

## 2 Toda chain: the periodic problem

Let us demonstrate the above formulas on the simplest Toda-chain model, which in the framework of the nonperturbative solutions corresponds to the  $4d$  *pure* gauge  $\mathcal{N} = 2$  supersymmetric Yang-Mills theory (or  $\mathcal{N} = 2$  SUSY gluodynamics) [23, 1]. The periodic problem in this model can be formulated in two different ways, which could be further deformed into two different directions. These deformations are hypothetically

related to the two different couplings of the 4d theory by adding the adjoint and fundamental matter  $\mathcal{N} = 2$  hypermultiplets correspondingly [4]-[8], [11, 12].

The Toda chain system is a simple system of particles where only the neighbor ones interact with the exponential potential and can be defined by the equations of motion

$$\frac{\partial \phi_i}{\partial t} = p_i \quad \frac{\partial p_i}{\partial t} = e^{\phi_{i+1}-\phi_i} - e^{\phi_i-\phi_{i-1}} \quad (30)$$

where one assumes (for the periodic problem with the “period”  $N_c$ ) that  $\phi_{i+N_c} = \phi_i$  and  $p_{i+N_c} = p_i$ . It is an integrable system, with  $N_c$  Poisson-commuting Hamiltonians,  $h_1^{TC} = \sum p_i$ ,  $h_2^{TC} = \sum (\frac{1}{2}p_i^2 + e^{\phi_i-\phi_{i-1}})$ , etc. As any finite-gap solution the periodic problem in Toda chain is formulated in terms of (the eigenvalues and the eigenfunctions of) two operators: the Lax operator (12)  $\mathcal{L}$  (or the auxiliary linear problem for (30))

$$\lambda \psi_n^\pm = \sum_k \mathcal{L}_{nk} \psi_k^\pm = e^{\frac{1}{2}(\phi_{n+1}-\phi_n)} \psi_{n+1}^\pm + p_n \psi_n^\pm + e^{\frac{1}{2}(\phi_n-\phi_{n-1})} \psi_{n-1}^\pm \quad (= \pm \frac{\partial}{\partial t} \psi_n^\pm) \quad (31)$$

and the second ( $\mathcal{A}$ -operator (16)) in this case can be chosen as a *monodromy* or shift operator in a discrete variable – the number of a particle

$$T\phi_n = \phi_{n+N_c} \quad Tp_n = p_{n+N_c} \quad T\psi_n = \psi_{n+N_c} \quad (32)$$

The common spectrum of these two operators <sup>5</sup>

$$\mathcal{L}\psi = \lambda\psi \quad T\psi = w\psi \quad [\mathcal{L}, T] = 0 \quad (33)$$

mean that there exists a relation between them  $\mathcal{P}(\mathcal{L}, T) = 0$  which can be strictly formulated in terms of spectral curve  $\Sigma$ :  $\mathcal{P}(\lambda, w) = 0$ . The generation function for these Hamiltonians can be written in terms of  $\mathcal{L}$  and  $T$  operators and the Toda chain possesses two essentially different formulations of this kind.

In the first version the Lax operator (31) is written in the basis of the  $T$ -operator eigenfunctions and becomes the  $N_c \times N_c$  matrix,

$$\mathcal{L}^{TC}(w) = \begin{pmatrix} p_1 & e^{\frac{1}{2}(\phi_2-\phi_1)} & 0 & we^{\frac{1}{2}(\phi_1-\phi_{N_c})} \\ e^{\frac{1}{2}(\phi_2-\phi_1)} & p_2 & e^{\frac{1}{2}(\phi_3-\phi_2)} & \dots & 0 \\ 0 & e^{\frac{1}{2}(\phi_3-\phi_2)} & p_3 & & 0 \\ & & \dots & & \\ \frac{1}{w}e^{\frac{1}{2}(\phi_1-\phi_{N_c})} & 0 & 0 & & p_{N_c} \end{pmatrix} \quad (34)$$

defined on the two-punctured sphere. Matrix (34) is almost three-diagonal as it follows from (31), the only extra nonzero elements appear in the off-diagonal corners exactly due to periodic conditions (32) reducing therefore naively infinite-dimensional matrix (31) to a finite-dimensional one depending on the spectral parameter  $w$ . The eigenvalues of the Lax operator (34) are defined from the spectral equation

$$\mathcal{P}(\lambda, w) = \det_{N_c \times N_c} (\mathcal{L}^{TC}(w) - \lambda) = 0 \quad (35)$$

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<sup>5</sup>Let us point out that we consider a *periodic* problem for the Toda chain when only the BA function can acquire a nontrivial factor under the action of the shift operator while the coordinates and momenta themselves are periodic. The *quasiperiodicity* of coordinates and momenta – when they acquire a nonzero shift – corresponds to the change of the coupling constant in the Toda chain Hamiltonians.



Substituting the explicit expression (34) into (35), one gets [39]:

$$w + \frac{1}{w} = 2P_{N_c}(\lambda) \quad (36)$$

or

$$y^2 = P_{N_c}^2(\lambda) - 1 \quad 2y = w - \frac{1}{w} \quad (37)$$

where  $P_{N_c}(\lambda)$  is a polynomial of degree  $N_c$ , with the coefficients being the Schur polynomials of the Hamiltonians

$$h_k = \sum_{i=1}^{N_c} p_i^k + \dots$$

$$P_{N_c}(\lambda) = \lambda^{N_c} + h_1 \lambda^{N_c-1} + \frac{1}{2}(h_2 - h_1^2) \lambda^{N_c-2} + \dots \quad (38)$$

The spectral equation depends only on the mutually Poisson-commuting combinations of the dynamical variables – the Hamiltonians or better action variables – parameterizing (a subspace in the) moduli space of the complex structures of the hyperelliptic curves  $\Sigma^{TC}$  of genus  $N_c - 1 = \text{rank} SU(N_c)$ .

An alternative description of the same system arises when one (before imposing periodic conditions) *solves* explicitly the auxiliary linear problem (31) which is a *second-order* difference equation. To do it one just rewrites (31) as

$$\psi_{i+1} = (\lambda - p_i) \psi_i - e^{\phi_i - \frac{\phi_{i+1} + \phi_{i-1}}{2}} \psi_{i-1} \quad (39)$$

or, since the space of solutions is 2-dimensional<sup>6</sup> (denoted by  $\psi^+$  and  $\psi^-$  in (31)), it can be rewritten as  $\tilde{\psi}_{i+1} = L_i^{TC}(\lambda) \tilde{\psi}_i$  where  $\tilde{\psi}_i$  is a set of two-vectors and  $L_i^{TC}$  – a chain of  $2 \times 2$  Lax matrices. After a simple "gauge" transformation these matrices can be written in the form [22]

$$L_i^{TC}(\lambda) = \begin{pmatrix} p_i + \lambda & e^{\phi_i} \\ e^{-\phi_i} & 0 \end{pmatrix}, \quad i = 1, \dots, N_c \quad (40)$$

This form is convenient to check integrable properties using the Hamiltonian language: the matrices (40) obey *quadratic*  $r$ -matrix Poisson bracket relations [40] (equivalent to  $\{\phi_i, p_j\} = \delta_{ij}$ )

$$\{L_i^{TC}(\lambda) \otimes L_j^{TC}(\lambda')\} = \delta_{ij} [r(\lambda - \lambda'), L_i^{TC}(\lambda) \otimes L_j^{TC}(\lambda')] \quad (41)$$

with the ( $i$ -independent!) numerical rational  $r$ -matrix  $r(\lambda) = \frac{1}{\lambda} \sum_{a=1}^3 \sigma_a \otimes \sigma^a$  satisfying the classical Yang-Baxter equation. As a consequence, the transfer matrix

$$T_{N_c}(\lambda) = \prod_{N_c \geq i \geq 1}^{\sim} L_i(\lambda) \quad (42)$$

satisfies the same Poisson-bracket relation

$$\{T_{N_c}(\lambda) \otimes T_{N_c}(\lambda')\} = [r(\lambda - \lambda'), T_{N_c}(\lambda) \otimes T_{N_c}(\lambda')] \quad (43)$$

and the integrals of motion of the Toda chain are generated by another form of spectral equation

$$\det_{2 \times 2} (T_{N_c}^{TC}(\lambda) - w) = w^2 - w \text{Tr} T_{N_c}^{TC}(\lambda) + \det T_{N_c}^{TC}(\lambda) = w^2 - w \text{Tr} T_{N_c}^{TC}(\lambda) + 1 = 0 \quad (44)$$

or

$$\mathcal{P}(\lambda, w) = \text{Tr} T_{N_c}^{TC}(\lambda) - w - \frac{1}{w} = 2P_{N_c}(\lambda) - w - \frac{1}{w} = 0 \quad (45)$$

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<sup>6</sup>The initial condition for the recursion relation (39) consists of two arbitrary functions, say,  $\psi_1$  and  $\psi_2$ .

(We used the fact that  $\det_{2 \times 2} L^{TC}(\lambda) = 1$  leads to  $\det_{2 \times 2} T_{N_c}^{TC}(\lambda) = 1$ .) The r.h.s. of (45) is a polynomial of degree  $N_c$  in  $\lambda$ , with the coefficients being the integrals of motion since

$$\begin{aligned} \{\text{Tr} T_{N_c}(\lambda), \text{Tr} T_{N_c}(\lambda')\} &= \text{Tr} \{T_{N_c}(\lambda) \otimes T_{N_c}(\lambda')\} = \\ &= \text{Tr} [r(\lambda - \lambda'), T_{N_c}(\lambda) \otimes T_{N_c}(\lambda')] = 0 \end{aligned} \quad (46)$$

The generating differential (21) for the Toda chain has the form

$$dS^{TC} = \lambda \frac{dw}{w} \quad (47)$$

Indeed, its derivatives over  $g$  moduli – the coefficients of the polynomial  $P_{N_c}(\lambda)$  in (38), (45)

$$\frac{\partial dS^{TC}}{\partial s_k} \equiv \left. \frac{\partial dS^{TC}}{\partial s_k} \right|_{\lambda=\text{const}} = \lambda d \frac{\partial}{\partial s_k} \log w = 2\lambda d \left( \frac{\frac{\partial P_{N_c}}{s_k}}{y} \right) \cong \frac{\lambda^{k+1} d\lambda}{y} \quad (48)$$

are (up to total derivatives which is denoted by  $\cong$ ) holomorphic differentials on curve (36), (37), (45). One should consider the derivatives over moduli as taken for fixed  $\lambda$  – see below more comments concerning this question. The explicit formulas for the prepotentials of the integrable system considered in this subsection will be presented in sect.3.

## 2.1 Integrable deformations of the Toda chain: coupling to matter

The  $N_c \times N_c$  matrix Lax operator (34) can be thought of as a "degenerate" case of the Lax operator for the  $GL(N_c)$  Calogero system [41]

$$\begin{aligned} \mathcal{L}^{Cal}(\xi) &= \left( \mathbf{pH} + \sum_{\alpha} F(\mathbf{q}\alpha|\xi) E_{\alpha} \right) = \\ &= \begin{pmatrix} p_1 & F(q_1 - q_2|\xi) & \dots & F(q_1 - q_{N_c}|\xi) \\ F(q_2 - q_1|\xi) & p_2 & \dots & F(q_2 - q_{N_c}|\xi) \\ & & \dots & \\ F(q_{N_c} - q_1|\xi) & F(q_{N_c} - q_2|\xi) & \dots & p_{N_c} \end{pmatrix} \end{aligned} \quad (49)$$

The matrix elements  $F(q|\xi) = m \frac{\sigma(q+\xi)}{\sigma(q)\sigma(\xi)} e^{\zeta(q)\xi}$  are expressed in terms of the Weierstrass elliptic functions and, thus, the Lax operator  $\mathcal{L}(\xi)$  is defined on the elliptic curve  $E(\tau)$  (complex torus with periods  $\omega, \omega'$  and modulus  $\tau = \frac{\omega'}{\omega}$ ). The Calogero coupling constant is  $m^2$ , where in the  $4d$  interpretation  $m$  plays the role of the mass of the adjoint matter  $\mathcal{N} = 2$  hypermultiplet breaking  $\mathcal{N} = 4$  SUSY down to  $\mathcal{N} = 2$  [4].

From (49) it follows that the spectral curve  $\Sigma^{Cal}$  for the  $GL(N_c)$  Calogero system is given by:

$$\det_{N_c \times N_c} (\mathcal{L}^{Cal}(\xi) - \lambda) = 0 \quad (50)$$

and is defined as covering of the elliptic curve  $E(\tau)$

$$y^2 = (x - e_1)(x - e_2)(x - e_3) \quad (51)$$

with the canonical holomorphic 1-differential

$$d\xi = 2 \frac{dx}{y} \quad (52)$$

The BPS masses  $\mathbf{a}$  and  $\mathbf{a}_D$  are now the periods of the generating 1-differential

$$dS^{Cal} \cong \lambda d\xi \quad (53)$$

along the non-contractable contours on  $\Sigma^{Cal}$ <sup>7</sup>. Integrability of the Calogero-Moser model can be described for example in terms of the following Poisson structure

$$\{\mathcal{L}(\xi) \otimes \mathcal{L}(\xi')\} = [\mathcal{R}_{12}^{Cal}(\xi, \xi'), \mathcal{L}(\xi) \otimes \mathbf{1}] - [\mathcal{R}_{21}^{Cal}(\xi, \xi'), \mathbf{1} \otimes \mathcal{L}(\xi')] \quad (54)$$

defined by *dynamical* elliptic  $\mathcal{R}$ -matrix [42], which guarantees the involution of the eigenvalues of matrix  $\mathcal{L}$ .

In order to recover the Toda-chain system, one takes the double-scaling limit [43], when  $m$  and  $-i\tau$  both go to infinity and

$$q_i - q_j = \frac{1}{2} [(i - j) \log m + (\phi_i - \phi_j)] \quad (55)$$

so that the dimensionless coupling  $\tau$  gets substituted by a dimensional parameter  $\Lambda^{N_c} \sim m^{N_c} e^{i\pi\tau}$ . In this limit, the elliptic curve  $E(\tau)$  degenerates into the (two-punctured) Riemann sphere with coordinate  $w = e^\xi e^{i\pi\tau}$  so that

$$dS^{Cal} \rightarrow dS^{TC} \cong \lambda \frac{dw}{w} \quad (56)$$

The Lax operator of the Calogero system turns into that of the  $N_c$ -periodic Toda chain (34):

$$\mathcal{L}^{Cal}(\xi) d\xi \rightarrow \mathcal{L}^{TC}(w) \frac{dw}{w} \quad (57)$$

and the spectral curve acquires the form (35). That is why the Calogero-Moser model can be considered as an elliptic deformation of the Toda chain. In contrast to the Toda case, (50) can *not* be rewritten in the form (36) and specific  $w$ -dependence of the spectral equation (35) is not preserved by embedding of Toda into Calogero-Moser particle system. However, the form (36) can be naturally preserved by the alternative deformation of the Toda-chain system when it is considered as (a particular case of) a spin-chain model.

In the simplest example of  $N_c = 2$ , the spectral curve  $\Sigma^{Cal}$  has genus 2. Indeed, in this particular case, eq.(50) turns into

$$\mathcal{P}(\lambda; x, y) = \lambda^2 - h_2 + \frac{g^2}{\omega^2} x = 0 \quad (58)$$

This equation says that with any value of  $x$  one associates two points of  $\Sigma^{Cal}$

$$\lambda = \pm \sqrt{h_2 - \frac{g^2}{\omega^2} x} \quad (59)$$

i.e. it describes  $\Sigma^{cal}$  as a double covering of the elliptic curve  $E(\tau)$  ramified at the points  $x = \left(\frac{\omega}{g}\right)^2 h_2$  and  $x = \infty$ . In fact,  $x = \left(\frac{\omega}{g}\right)^2 h_2$  corresponds to a *pair* of points on  $E(\tau)$  distinguished by the sign of  $y$ . This would be true for  $x = \infty$  as well, but  $x = \infty$  is one of the branch points in our parameterization (51) of  $E(\tau)$ . Thus, the *two* cuts between  $x = \left(\frac{\omega}{g}\right)^2 h_2$  and  $x = \infty$  on every sheet of  $E(\tau)$  touching at the common end at

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<sup>7</sup>Let us point out that the curve (50) has genus  $g = N_c$  (while in general the genus of the curve defined by  $N_c \times N_c$  matrix grows as  $N_c^2$ ) but the integrable system is still  $2(N_c - 1)$ -dimensional, i.e. one of the periods of (53) vanishes identically due to the symmetries of the curve (50) and there are only  $N_c - 1 = g - 1$  independent integrals of motion. This is a particular case of the Prym manifold considered above when one restricts himself only to those contours which are trivial after projection to a "bare" curve.

$x = \infty$  become effectively a *single* cut between  $\left(\left(\frac{\omega}{g}\right)^2 h_2, +\right)$  and  $\left(\left(\frac{\omega}{g}\right)^2 h_2, -\right)$ . Therefore, we can consider the spectral curve  $\Sigma^{Cal}$  as two tori  $E(\tau)$  glued along one cut, i.e.  $\Sigma_{N_c=2}^{Cal}$  has genus 2. It turns out to be a hyperelliptic curve (for  $N_c = 2$  only!) after substituting in (58)  $x$  from the second equation to the first one.

Two holomorphic 1-differentials on  $\Sigma^{Cal}$  ( $g = N_c = 2$ ) can be chosen to be

$$v = \frac{dx}{y} \sim \frac{\lambda d\lambda}{y} \quad V = \frac{dx}{y\lambda} \sim \frac{d\lambda}{y} \quad (60)$$

so that

$$dS \cong \lambda d\xi = \sqrt{h_2 - \frac{g^2}{\omega^2} \wp(\xi)} d\xi = \frac{dx}{y} \sqrt{h_2 - \frac{g^2}{\omega^2} x} \quad (61)$$

It is easy to check the basic property (21):

$$\frac{\partial dS}{\partial h_2} \cong \frac{1}{2} \frac{dx}{y\lambda} \quad (62)$$

The fact that only one of two holomorphic 1-differentials (60) appears at the r.h.s. is related to their different parity with respect to the  $\mathbf{Z}_2 \otimes \mathbf{Z}_2$  symmetry of  $\Sigma^{Cal}$ :  $y \rightarrow -y$  and  $\lambda \rightarrow -\lambda$ . Since  $dS$  has certain parity, its integrals along the two of four elementary non-contractable cycles on  $\Sigma^{Cal}$  automatically vanish leaving only two non-vanishing quantities  $a$  and  $a_D$ , as necessary for the 4d interpretation. Moreover, two rest nonzero periods can be defined in terms of the genus 1 "reduced" curve

$$Y^2 = (y\lambda)^2 = \left(h_2 - \frac{g^2}{\omega^2} x\right) \prod_{a=1}^3 (x - e_a), \quad (63)$$

equipped with  $dS \cong \left(h_2 - \frac{g^2}{\omega^2} x\right) \frac{dx}{Y}$ . Since for this curve  $x = \infty$  is no more a ramification point,  $dS$  has simple poles when  $x = \infty$  (on both sheets of  $\Sigma_{reduced}^{Cal}$ ) with the residues  $\pm \frac{g}{\omega} \sim \pm m$ .

The opposite limit of the Calogero-Moser system with vanishing coupling constant  $g^2 \sim m^2 \rightarrow 0$  corresponds to the  $\mathcal{N} = 4$  SUSY Yang-Mills theory with identically vanishing  $\beta$ -function. The corresponding integrable system is a collection of *free* particles and the generating differential  $dS \cong \sqrt{h_2} \cdot d\xi$  is just a *holomorphic* differential on  $E(\tau)$ .

Now, let us turn to another deformation of the Toda chain corresponding to the coupling of the  $\mathcal{N} = 2$  SYM theory to the fundamental matter. According to [21, 24], the spectral curves for the  $\mathcal{N} = 2$  SQCD with any  $N_f < 2N_c$  have the same form as (33) with a less trivial monodromy matrix with the invariants

$$\text{Tr } T_{N_c}(\lambda) \equiv 2P_{N_c}(\lambda) = 2P_{N_c}^{(0)}(\lambda) + R_{N_c-1}(\lambda), \quad \det T_{N_c}(\lambda) = Q_{N_f}(\lambda), \quad (64)$$

and  $Q_{N_f}(\lambda)$  and  $R_{N_c-1}(\lambda)$  are certain  *$h$ -independent* polynomials of  $\lambda$ .

A natural proposal is to look at the orthogonal to the previous generalization of the Toda chain, i.e. deform eqs.(41)-(46) preserving the Poisson brackets

$$\{L(\lambda) \otimes L(\lambda')\} = [r(\lambda - \lambda'), L(\lambda) \otimes L(\lambda')], \quad (65)$$

$$\{T_{N_c}(\lambda) \otimes T_{N_c}(\lambda')\} = [r(\lambda - \lambda'), T_{N_c}(\lambda) \otimes T_{N_c}(\lambda')],$$

and, thus, the possibility to build a monodromy matrix  $T(\lambda)$  by multiplication of  $L_i(\lambda)$ 's. The full spectral curve for the periodic spin chain is still given by:

$$\det(T_{N_c}(\lambda) - w) = 0, \quad (66)$$

but since in general  $\det T_{N_c}(\lambda) = \prod_{i=1}^{N_c} \det L(\lambda - \lambda_i) \neq 1$  equation (66) acquires the form

$$w + \frac{\det T_{N_c}(\lambda)}{w} = \text{Tr} T_{N_c}(\lambda), \quad (67)$$

or

$$W + \frac{1}{W} = \frac{\text{Tr} T_{N_c}(\lambda)}{\sqrt{\det T_{N_c}(\lambda)}} \equiv \frac{2P_{N_c}(\lambda)}{\sqrt{Q_{N_f}(\lambda)}} \quad (68)$$

while generating 1-form is now

$$dS = \lambda \frac{dW}{W} \quad (69)$$

The r.h.s. of the equations (67), (68) contain the dynamical variables of the spin system only in the special combinations – its Hamiltonians (which are all in involution, i.e. Poisson-commuting). The explicit examples can be found in [11, 12].

In this picture the rational  $XXX$  spin chain literally corresponds to a  $N_f < 2N_c$   $\mathcal{N} = 2$  SUSY QCD while the conformal  $N_f = 2N_c$  case when an additional dimensionless parameter appears is hypothetically described by the  $XYZ$  chain with the Hamiltonian structure given by elliptic Sklyanin algebra [40] (see [12] for detailed analysis of this case where however still a lot of open questions exist).

## 2.2 Symplectic structure of the finite-gap systems

Now let us turn to the discussion of a more subtle point – why the generating 1-form (21) indeed describes an integrable system. To do this I consider, first, the simplest definition of the generating 1-form. According to this definition the generating form (21) defines the symplectic structure on the phase space of the finite-gap solutions. This symplectic structure was introduced in [29] and recently proposed in [13] as coming directly from the symplectic form on the space of all the solutions to the hierarchy. Below I give the most simple and straightforward proof of this fact as presented in [14, 15] which is supplied by concrete examples having direct relation to the integrable systems arising in the formulation of nonperturbative results in quantum theories. In addition we will discuss the relation of generating differential to the duality transformation in nonperturbative  $c < 1$  string theory.

To prove that (47) is a generating one-form of the whole hierarchy one starts with the variation of the generating function

$$S(\Sigma, \gamma) = \sum_i \int^{\gamma_i} dS = \sum_i \int^{\gamma_i} E dp \quad (70)$$

(where  $dE$  and  $dp$  ( $= d\lambda$  and  $= \frac{dw}{w}$ ) in the particular case above) are two meromorphic differentials (with fixed periods: for example for the hyperelliptic coordinate  $\lambda$  all  $\oint d\lambda = 0$ ) on a spectral curve  $\Sigma$  and  $\gamma$  is the divisor of the solution (i.e. the set of points – with their multiplicities – the poles of the BA function)). Taking the first variation <sup>8</sup>, one gets

$$\delta S = \sum_i (E dp)(\gamma_i) + \sum_i \int^{\gamma_i} \delta E dp \quad \delta^2 S = \delta \left( \sum_i (E dp)(\gamma_i) \right) + \sum_i (\delta E dp)(\gamma_i) \quad (71)$$

---

<sup>8</sup>The total variation, which includes the variation of the moduli of complex structure of the curve is considered. To compare the differentials on two curves with different complex structure one should introduce the connection, which will be chosen below as satisfying the condition  $\delta_{moduli} p = 0$ , i.e. such that function  $p$  is covariantly constant (one can compare it with [13], where the role of covariantly constant function is played by function  $E$ ).

From  $\delta^2 S = 0$  it follows that

$$\varpi = \delta E \wedge \delta p = \delta \left( \sum_i (Edp)(\gamma_i) \right) = - \sum_i (\delta Edp)(\gamma_i) \quad (72)$$

Now, the variation  $\delta E$  (for constant  $p$ ) follows from the Lax equation (auxiliary linear problem)

$$\frac{\partial}{\partial t} \psi = \mathcal{L} \psi (= E\psi) \quad (73)$$

so that

$$\delta E = \frac{\langle \psi^\dagger \delta \mathcal{L} \psi \rangle}{\langle \psi^\dagger \psi \rangle} \quad (74)$$

and one concludes that

$$\varpi = - \langle \delta \mathcal{L} \sum_i \left( dp \frac{\psi^\dagger \psi}{\langle \psi^\dagger \psi \rangle} \right) (\gamma_i) \rangle \quad (75)$$

Let us turn to several important examples.

**KP/KdV.** In the KP-case the equation (73) looks as

$$\frac{\partial}{\partial t} \psi = \left( \frac{\partial^2}{\partial x^2} + u \right) \psi (= E\psi) \quad (76)$$

therefore the equation (75) implied by  $\langle \psi^\dagger \psi \rangle = \int_{dx} \psi^\dagger(x, P) \psi(x, P)$  and  $\delta \mathcal{L} = \delta u(x)$  gives

$$\varpi = - \int_{dx} \delta u(x) \sum_i \left( \frac{dp}{\langle \psi^\dagger \psi \rangle} \psi^\dagger(x) \psi(x) \right) (\gamma_i) \quad (77)$$

The differential  $d\Omega = \frac{dp}{\langle \psi^\dagger \psi \rangle} \psi^\dagger(x) \psi(x)$  is holomorphic on  $\Sigma$  except for the "infinity" point  $P_0$  where it has zero residue [44]. Vanishing of the sum of residues of its "special" variation <sup>9</sup>

$$\tilde{\delta} \left( \text{res}_{P_0} d\Omega + \sum_i \text{res}_{\gamma_i} d\Omega \right) = 0 \quad (78)$$

can be rewritten as

$$\delta v(x) + \sum_i d\Omega(\gamma_i) = 0 \quad (79)$$

where  $v(x)$  is a "residue" of the BA function at the point  $P_0$  obeying  $v'(x) = u(x)$ . Substituting (79) into (75) one gets

$$\varpi = \int_{dx} \delta u(x) \int_{dx'} \delta u(x') \quad (80)$$

or the *first* symplectic structure of the KdV equation.

**Toda chain/lattice.** (The case directly related to the  $\mathcal{N} = 2$  pure SYM theory). One has

$$\langle \psi^\dagger \psi \rangle = \sum_n \psi_n^+(P) \psi_n^-(P)$$

and the Lax equation acquires the form (31) where  $t = t_+ + t_-$  and  $t_1 = t_+ - t_-$  is the first time of the Toda chain, so that

$$\delta \lambda = \frac{\sum_k \psi_k^+ \delta p_k \psi_k^-}{\langle \psi^+ \psi^- \rangle} \quad (81)$$

and (75) becomes

$$\varpi = - \sum_k \delta p_k \sum_i \left( \frac{dp}{\langle \psi^+ \psi^- \rangle} \psi_k^+ \psi_k^- \right) (\gamma_i) \quad (82)$$

---

<sup>9</sup>It should be pointed out that the variation  $\tilde{\delta}$  corresponds to a rather specific situation when one shifts only  $\psi$  keeping  $\psi^\dagger$  fixed.

Thus, to get

$$\varpi = \sum_k \delta p_k \wedge \delta q_k \quad (83)$$

one has to prove

$$\sum_i \left( \frac{dp}{\langle \psi^+ \psi^- \rangle} \psi_k^+ \psi_k^- \right) (\gamma_i) = \delta q_k \quad (84)$$

To do this one considers again

$$\tilde{\delta} \left( \text{res}_{P_+} + \text{res}_{P_-} + \sum_i \text{res}_{\gamma_i} \right) d\Omega_n = 0 \quad d\Omega_n = \frac{dp}{\langle \psi^+ \psi^- \rangle} \psi_n^+ \psi_n^- \quad (85)$$

where the first two terms for  $\psi_n^\pm \underset{\lambda \rightarrow \lambda(P_\pm)}{\sim} e^{\pm q_n} \lambda^{\pm n} (1 + \mathcal{O}(\lambda^{-1}))$  satisfying two "shifted" equations (31) (with  $\tilde{q}_n$  and  $q_n$  correspondingly) give  $\delta q_n = \tilde{q}_n - q_n$  while the rest – the l.h.s. of (84).

**Calogero-Moser system.** Introducing the "standard"  $dE$  and  $dp$  on the curve  $\Sigma$  (50) with the 1-form (53) where  $dp = d\xi$  is holomorphic on torus  $\oint_A dp = \omega$ ,  $\oint_B dp = \omega'$  and  $E = \lambda$  has  $N_c - 1$  poles with *residue* = 1 and 1 pole with *residue* =  $-(N_c - 1)$ , the BA function is defined by [41]

$$\mathcal{L}^{Cal}(\xi) \mathbf{a} = \lambda \mathbf{a} \quad (86)$$

with the essential singularities

$$a_i \underset{E=E_+}{\sim} e^{x_i \zeta(\xi)} (1 + \mathcal{O}(\xi)) \quad a_i \underset{E \neq E_+}{\sim} e^{x_i \zeta(\xi)} \left( -\frac{1}{n-1} + \mathcal{O}(\xi) \right) \quad (87)$$

and (independent of dynamical variables) poles  $\gamma$ . Hence, similarly to the above case for the eq. (75) one has  $\langle \psi^\dagger \psi \rangle = \sum_i a_i^\dagger(P) a_i(P)$ ,  $\delta \mathcal{L}^{Cal} = \frac{\sum_i a_i^\dagger(P) \delta p_i a_i(P)}{\sum_i a_i^\dagger(P) a_i(P)}$  so that the expression (75) acquires the form

$$\varpi = - \sum_k \delta p_k \sum_i \left( \frac{d\xi}{\langle a^\dagger a \rangle} a_k^\dagger a_k \right) (\gamma_i) \quad (88)$$

and the residue formula

$$\tilde{\delta} \left( \sum_{P_j: p=0} \text{res}_{P_j} + \sum_i \text{res}_{\gamma_i} \right) d\Omega_k = 0 \quad d\Omega_k = \frac{d\xi}{\langle a^\dagger a \rangle} a_k^\dagger a_k \quad (89)$$

where the first sum is over all "infinities"  $p = 0$  at each sheet of the cover (50). After variation and using (87) it gives again

$$\varpi = \sum_k \delta p_k \wedge \delta x_k \quad (90)$$

The general proof of the more cumbersome analog of the above derivation can be found in [13]. The explicit form of the obtained formulas looks rather nontrivial, hence, to illustrate their existence let us, finally, demonstrate the existence of (78), (85) and (89) for the 1-gap solution. Let

$$\psi = e^{x\zeta(z)} \frac{\sigma(x-z+\kappa)}{\sigma(x+\kappa)\sigma(z-\kappa)} \quad \psi^\dagger = e^{-x\zeta(z)} \frac{\sigma(x+z+\kappa)}{\sigma(x+\kappa)\sigma(z+\kappa)} \quad (91)$$

be solutions to

$$(\partial^2 + u)\psi = (\partial^2 - 2\wp(x+\kappa))\psi = \wp(z)\psi \quad (92)$$

Then

$$\begin{aligned} \psi^\dagger \psi &= \frac{\sigma(x-z+\kappa)\sigma(x+z+\kappa)}{\sigma^2(x+\kappa)\sigma(z-\kappa)\sigma(z+\kappa)} = \frac{\sigma^2(z)}{\sigma(z+\kappa)\sigma(z-\kappa)} (\wp(z) - \wp(x+\kappa)) \\ \langle \psi^\dagger \psi \rangle &= \frac{\sigma^2(z)}{\sigma(z+\kappa)\sigma(z-\kappa)} (\wp(z) - \langle \wp(x+\kappa) \rangle) \end{aligned} \quad (93)$$

and let us take the average over a period  $2\tilde{\omega}$  to be  $\langle \wp(x + \kappa) \rangle = 2\tilde{\eta}$ . Also

$$\begin{aligned} dp &= d(\zeta(z) + \log \sigma(2\tilde{\omega} - z + \kappa) - \log \sigma(\kappa - z)) = -dz(\wp(z) + \zeta(2\tilde{\omega} - z + \kappa) - \zeta(\kappa - z)) = \\ &= -dz(\wp(z) - 2\tilde{\eta}) \end{aligned} \quad (94)$$

and

$$\begin{aligned} \frac{dp}{\langle \psi^\dagger \psi \rangle} &= dz \frac{\sigma(z + \kappa)\sigma(z - \kappa)}{\sigma^2(z)} \\ d\Omega &= \frac{dp}{\langle \psi^\dagger \psi \rangle} \psi^\dagger \psi = dz \frac{\sigma(x + z + \kappa)\sigma(x + z - \kappa)}{\sigma^2(z)\sigma^2(x + \kappa)} = dz(\wp(z) - \wp(x + \kappa)) \end{aligned} \quad (95)$$

Now, the variation  $\tilde{\delta}$  explicitly looks as

$$\begin{aligned} \tilde{\delta}d\Omega &\equiv \frac{dp}{\langle \psi^\dagger \psi \rangle} \psi^\dagger \psi_{\kappa + \delta\kappa} = dz \frac{\sigma(z + \kappa)\sigma(z - \kappa)\sigma(x + z + \kappa)\sigma(x - z + \kappa + \delta\kappa)}{\sigma^2(z)\sigma(x + \kappa)\sigma(z + \kappa)\sigma(x + \kappa + \delta\kappa)\sigma(z - \kappa - \delta\kappa)} = \\ &= dz \frac{\sigma(x + \kappa + z)\sigma(x + \kappa - z)}{\sigma^2(z)\sigma^2(x + \kappa)} [1 + \delta\kappa(\zeta(x - z + \kappa) + \zeta(z - \kappa) - \zeta(x + \kappa)) + \mathcal{O}((\delta\kappa)^2)] \\ &= dz(\wp(z) - \wp(x + \kappa)) [1 + \delta\kappa(\zeta(x - z + \kappa) + \zeta(z - \kappa) - \zeta(x + \kappa)) + \mathcal{O}((\delta\kappa)^2)] \end{aligned} \quad (96)$$

It is easy to see that (96) has non-zero residues at  $z = 0$  and  $z = \kappa$  (the residue at  $z = x + \kappa$  is suppressed by  $\wp(z) - \wp(x + \kappa)$ ). They give

$$\begin{aligned} \text{res}_{z=0} \delta d\Omega &\sim \delta\kappa \oint_{z \rightarrow 0} dz \wp(z) (\zeta(x - z + \kappa) + \zeta(z - \kappa)) \sim \delta\kappa \oint_{z \rightarrow 0} \zeta(z) d(\zeta(x - z + \kappa) + \\ &+ \zeta(z - \kappa)) \sim \delta\kappa (\wp(x + \kappa) + \wp(\kappa)) \sim \delta(\zeta(x + \kappa) + \zeta(\kappa)) \equiv \delta v(x) \end{aligned} \quad (97)$$

and

$$\text{res}_{z=\kappa} \delta d\Omega = \delta\kappa (\wp(\kappa) - \wp(x + \kappa)) = d\Omega(\kappa) \quad (98)$$

which follows from the comparison to (95). Thus, we have checked the existence of (78) and (79) in the simplest possible example by explicit computation.

Finally, the quantization of the symplectic structure (72), at least in the cases when it is known, corresponds to the complete formulation of the effective theory. The corresponding generating function  $S = \sum_k \int^{\gamma_k} dS$  defines the duality transformation between the dual integrable systems (see [33, 34]). The formulation of the complete effective theory hypothetically corresponds to the exact answer for the full generating function  $\log \mathcal{T} = \log \mathcal{T}_0 + \log \mathcal{T}_\theta \equiv \mathcal{F} + \log \mathcal{T}_\theta$ , including also the deformation of the oscillating part corresponding to *massive* states in the spectrum.

## 2.3 Comments on main definitions

Let us now make some comments on the main formulas (21) etc we used above. In fact these formulas need to be defined in a more strict way since they imply taking derivatives of various objects (holomorphic and meromorphic differentials on complex curves) over moduli and *a priori* this is not a very well-defined procedure. In most of concrete cases the hyperelliptic curves were used (i.e. the curves which possess a special coordinate  $\lambda$  or differential  $d\lambda$  with zero periods along any cycles), and by derivatives over ("hyperelliptic") moduli – or over the ramification points one meant

$$\frac{\partial}{\partial h_k} dS \equiv \frac{\partial}{\partial h_k} dS(\lambda, h_k) \Big|_{\lambda=\text{const}} \quad (99)$$

i.e.  $\frac{\partial}{\partial h_k} d\lambda = 0$ . Now, one should consider at least two arising questions:



- Why this definition is reasonable for hyperelliptic Riemann surfaces (e.g. (36), (58), (68))?
- What should be done instead, when there is no specific coordinate or differential like  $d\lambda$ ?

In this subsection I would try at least partially to answer to this question. These comments arose as a result of discussions with A. Levin and his many clear explanations.

Consider the definition of an integrable system in spirit of [29]

$$\frac{\partial}{\partial h_k} dS \cong dv_k \quad (100)$$

where  $dS$  is generating differential and  $dv_k$  are *some* holomorphic differentials. We will try to demonstrate how the generating differential (21) can be constructed for a generic integrable system.

Consider, first, the purely holomorphic case (appearing for example in the framework of the Hitchin systems [38]). The most general construction of the differentials of the (21) type is based on the existence of a differentials with  $g - 1$  *double* zeroes on a curve  $\Sigma_g$  of genus  $g$ . These differentials look like

$$dS(P) \sim \sum_{j=1}^g \frac{\partial \Theta_*(\mathbf{0})}{\partial A_j} d\omega_j(P) \equiv H_*(P) \equiv \nu_*^2(P) \quad (101)$$

where  $\mathbf{A}(P) = \int^P d\boldsymbol{\omega}$  is the Abel map into Jacobian  $\text{Jac}(\Sigma_g)$ ,  $\Theta$  – Riemann theta-function [54, 55],  $\nu_*(P)$  is a section of  $K^{\frac{1}{2}}$  corresponding to an even characteristic  $*$  having simple zeroes at points  $R_1, \dots, R_{g-1}$ . On a genus  $g$  curve one has  $2^{2g-1} + 2^{g-1}$  even and  $2^{2g-1} - 2^{g-1}$  odd characteristics.

Now one should define the co-ordinates  $\{h_k\}$  – *locally* the directions in moduli space such that the derivatives in this direction give holomorphic differentials. The correspondence can be established as

$$1 + \epsilon \frac{\partial}{\partial h_k} \leftrightarrow 1 + \epsilon L_{-2}^{(k)} \quad (102)$$

$$L_{-2}^{(k)} = \frac{1}{\xi_k} \frac{\partial}{\partial \xi_k}$$

where  $\xi_k$  is a co-ordinate in the vicinity of a point  $R_k$ . They obviously commute

$$[L_{-2}^{(j)}, L_{-2}^{(k)}] = 0 \quad (103)$$

An easy way to check the correspondence is to consider the following basis in the space of holomorphic 1-differentials on  $\Sigma_g$  [53]:

$$dv_k \underset{P \rightarrow R_k}{\sim} d\xi_k \underset{P \rightarrow R_j}{+} \delta_{jk} \xi_j d\xi_j$$

$$dv_g = dS \underset{P \rightarrow R_j}{\sim} \xi_j^2 d\xi_j \quad (104)$$

$$j \neq k \in \{1, \dots, g-1\}$$

It is possible to write down explicit formulas for the differentials  $dv_k$ . To do this one should first write down a section of  $K^{\frac{1}{2}}$  with the only simple pole at  $R_j$ . It has the following form

$$\psi_j(P) \sim \frac{\frac{\partial}{\partial A_i} \Theta_*(\mathbf{A}(P) - \mathbf{A}(R_j))}{\frac{\partial}{\partial A_i} \Theta_*(\mathbf{0}) E(P, R_j)} \quad (105)$$

$$\psi_j(P) \underset{P \rightarrow R_j}{\sim} \frac{\sqrt{d\xi_j}}{\xi_j} + \dots$$

where we used explicitly the Prime form

$$E(P, P_0) = \frac{\Theta_*(\mathbf{A}(P) - \mathbf{A}(P_0))}{\nu(P)\nu(P_0)} \quad (106)$$

Then, obviously

$$dv_k(P) = \psi_k(P)\nu_*(P) \quad (107)$$

Now, to check (21) one acts by generators (102) to the  $dS = dv_g$ . The result obviously is

$$\begin{aligned} \left(1 + \epsilon \frac{\partial}{\partial h_i}\right) dS &= \left(1 + \epsilon L_{-2}^{(i)}\right) dS = \\ &= \left(1 + \epsilon \frac{1}{\xi_i} \frac{\partial}{\partial \xi_i}\right) (\xi_i^2 d\xi_i + \dots) = \left(\xi_i + \frac{\epsilon}{\xi_i}\right)^2 d\left(\xi_i + \frac{\epsilon}{\xi_i}\right) = \\ &= \xi_i^2 d\xi_i + \epsilon d\xi_i + \mathcal{O}(\epsilon^2) + \dots = dS + \epsilon dv_k + \mathcal{O}(\epsilon^2) \end{aligned} \quad (108)$$

i.e. indeed the equality (21). To get  $2g$ -dimensional integrable system one could add as a parameter the normalization of  $dS$ , i.e.

$$dS = h_g dv_g \quad (109)$$

The above considerations can be almost literally transformed to the more interesting meromorphic case. The corresponding formulas can be extracted from [54].

## 2.4 Whitham equations

Now, let us turn to another definition of the generating 1-form (47). This definition goes back to the general approach to construction of the effective actions which is known as the Bogolyubov-Whitham averaging method (see [27, 36, 45, 46] for a comprehensive review and references). Though the Whitham dynamics describes the commutative flows on the moduli spaces, averaging over the Jacobian – the fast part of the theory, its explicit formulation is most simple and natural in terms of connections on spectral curves [36, 45]. The convenience of the Whitham language is caused, in part, by the fact that the previous consideration does not lead to any natural definition of the prepotential  $\mathcal{F}$ . The Whitham dynamics allows one to define generating differential (21) in such a way that a natural identification of (27) with the *logarithm of  $\tau$ -function* of the Whitham hierarchy appears.

The Whitham equations determine a flat coordinate system on some (finite-dimensional, complex) space [36], which usually appears in interesting examples as the space of complex structures of a Riemann surface, associated to a finite-gap solution to the equations of KP/Toda type. The most part of interesting solutions to the Whitham hierarchy is related to the "modulation" of parameters of the finite-gap solutions of integrable systems of KP/Toda type. The KP  $\tau$ -function associated with a given spectral curve  $\Sigma_g$  is

$$\mathcal{T}\{t_i\} = e^{\sum t_i \gamma_{ij} t_j} \Theta\left(\Phi_0 + \sum t_i \mathbf{k}_i\right) \quad \mathbf{k}_i = \oint_{\mathbf{B}} d\Omega_i \quad (110)$$

where  $\Theta$  is a Riemann theta-function [54, 55] – a function on the Jacobian  $\text{Jac}(\Sigma_g)$  and  $d\Omega_i$  are meromorphic 1-differentials with poles of the order  $i+1$  at a marked point  $z_0$ . They are completely specified by the normalization relations

$$\oint_{\mathbf{A}} d\Omega_i = 0 \quad (111)$$

and the asymptotic behavior

$$d\Omega_i = (\xi^{-i-1} + o(\xi)) d\xi \quad (112)$$

where  $\xi$  is a local coordinate in the vicinity of  $z_0$ . The moduli  $\{u_\alpha\}$  of spectral curve are invariants of KP flows,

$$\frac{\partial u_\alpha}{\partial t_i} = 0, \quad (113)$$

However, the moduli become dependent on  $t_i$  after the "modulation" defined by the Whitham equations which for a particular choice of the coordinate on  $\Sigma$  acquire the most simple form

$$\frac{\partial d\Omega_i(z)}{\partial t_j} = \frac{\partial d\Omega_j(z)}{\partial t_i}. \quad (114)$$

and imply that

$$d\Omega_i(z) = \frac{\partial dS(z)}{\partial t_i} \quad (115)$$

and the equations for moduli, following from (114), are:

$$\frac{\partial u_\alpha}{\partial t_i} = v_{ij}^{\alpha\beta}(u) \frac{\partial u_\beta}{\partial t_j} \quad (116)$$

with some (in general complicated) functions  $v_{ij}^{\alpha\beta}$ .

From the point of view of subsect.2.3 the Whitham equations relate two different set of (flat) coordinates on moduli space. One is generated by "averaged" KP/Toda flows and is given by  $L_{-n}^0$  – the singular Virasoro generators in the picture  $P_0$  corresponding to the KP hierarchy from geometrical point of view. The other are coordinates induced by  $L_{-2}^\alpha$  – in the particular points  $R_\alpha$  on the curve  $\Sigma$  which are branch points (or Riemann invariants) in the hyperelliptic case.

In the KdV (and Toda-chain) case all the spectral curves are hyperelliptic, and for the KdV  $i$  takes only odd values  $i = 2j + 1$ , so that

$$d\Omega_{2j+1}(z) = \frac{\mathcal{P}_{j+g}(z)}{y(z)} dz, \quad (117)$$

the coefficients of the polynomials  $\mathcal{P}_j$  being fixed by normalization conditions (111), (112) (one usually takes  $z_0 = \infty$  and the local parameter in the vicinity of this point is  $\xi = z^{-1/2}$ ). In this case the equations (116) can be diagonalized if the co-ordinates  $\{u_\alpha\}$  on the moduli space are taken to be the ramification points:

$$v_{ij}^{\alpha\beta}(u) = \delta^{\alpha\beta} \left. \frac{d\Omega_i(z)}{d\Omega_j(z)} \right|_{z=u_\alpha} \quad (118)$$

Finally, the differential  $dS(z)$  (115) can be constructed for any finite-gap solution [29] and it *coincides* with the generating 1-form (47). The equality

$$\frac{\partial \mathcal{F}}{\partial \mathbf{a}} = \mathbf{a}_D \quad \mathbf{a} = \oint_{\mathbf{A}} dS \quad \mathbf{a}_D = \oint_{\mathbf{B}} dS \quad (119)$$

defines  $\tau$ -function of the Whitham hierarchy  $\mathcal{F} = \log \mathcal{T}_{Whitham}$  [36, 45]. The exact answer for the partition function  $\log \mathcal{T} = \log \mathcal{T}_0 + \log \mathcal{T}_\theta \equiv \mathcal{F} + \log \mathcal{T}_\theta$  should also include the deformation of the oscillating part, corresponding to the *massive* excitations. Below, the explicit examples of the Whitham solutions will be considered.

Now let us demonstrate that the higher genus Riemann surfaces (already in the elliptic case) give rise to nonperturbative formulation of physically less trivial theories. In contrast to the previous example Whitham

times will be nontrivially related to the moduli of the curve. The elliptic solution to the KdV is

$$\begin{aligned} U(t_1, t_3, \dots | u) &= \frac{\partial^2}{\partial t_1^2} \log \mathcal{T}(t_1, t_3, \dots | u) = \\ &= U_0 \wp(k_1 t_1 + k_3 t_3 + \dots + \Phi_0 | \omega, \omega') + \frac{u}{3} \end{aligned} \quad (120)$$

and

$$\begin{aligned} dp &\equiv d\Omega_1(z) = \frac{z - \alpha(u)}{y(z)} dz, \\ dQ &\equiv d\Omega_3(z) = \frac{z^2 - \frac{1}{2}uz - \beta(u)}{y(z)} dz. \end{aligned} \quad (121)$$

Normalization conditions (111) prescribe that

$$\alpha(u) = \frac{\oint_A \frac{z dz}{y(z)}}{\oint_A \frac{dz}{y(z)}} \quad \beta(u) = \frac{\oint_A \frac{(z^2 - \frac{1}{2}uz) dz}{y(z)}}{\oint_A \frac{dz}{y(z)}}. \quad (122)$$

The simplest elliptic example is the first Gurevich-Pitaevsky (GP) solution [47] with the underlying spectral curve

$$y^2 = (z^2 - 1)(z - u) \quad (123)$$

specified by a requirement that all branching points except for  $z = u$  are *fixed* and do not obey Whitham deformation. It is easy to see that by change of variables  $z = u + \lambda^2$  and  $y \rightarrow y\lambda$  the curve (123) can be written as (a particular  $N_c = 2$  case) of the Toda-chain curve (37)  $y^2 = (\lambda^2 + u)^2 - 1$  c  $P_{N_c=2}(\lambda) = \lambda^2 + h_2$  i.e.  $h_2 \equiv u$ . The generating differential (47), (115), corresponding to (123) is given by [1]

$$dS(z) = \left( t_1 + t_3 \left( z + \frac{1}{2}u \right) + \dots \right) \frac{z - u}{y(z)} dz \quad \underset{\{t_{k>1}=0\}}{=} \quad t_1 \frac{z - u}{y(z)} dz \quad (124)$$

and it produces the simplest solution to (114) coming from the elliptic curve. From (124) one derives:

$$\begin{aligned} \frac{\partial dS(z)}{\partial t_1} &= \left( z - u - \left( \frac{1}{2}t_1 + ut_3 \right) \frac{\partial u}{\partial t_1} \right) \frac{dz}{y(z)}, \\ \frac{\partial dS(z)}{\partial t_3} &= \left( z^2 - \frac{1}{2}uz - \frac{1}{2}u^2 - \left( \frac{1}{2}t_1 + ut_3 \right) \frac{\partial u}{\partial t_3} \right) \frac{dz}{y(z)}, \\ &\dots, \end{aligned} \quad (125)$$

and comparison with explicit expressions (121) implies:

$$\begin{aligned} \left( \frac{1}{2}t_1 + ut_3 \right) \frac{\partial u}{\partial t_1} &= \alpha(u) - u, \\ \left( \frac{1}{2}t_1 + ut_3 \right) \frac{\partial u}{\partial t_3} &= \beta(u) - \frac{1}{2}u^2. \end{aligned} \quad (126)$$

In other words, this construction provides the first GP solution to the Whitham equation

$$\frac{\partial u}{\partial t_3} = v_{31}(u) \frac{\partial u}{\partial t_1}, \quad (127)$$

with

$$v_{31}(u) = \frac{\beta(u) - \frac{1}{2}u^2}{\alpha(u) - u} = \frac{d\Omega_3(z)}{d\Omega_1(z)} \Big|_{z=u}, \quad (128)$$

which can be expressed through elliptic integrals [47]. More detailed analysis can be found in [9].

### 3 Prepotential of the Effective Theory

In the previous section the prepotential  $\mathcal{F}$  was identified with the logarithm of the  $\tau$ -function of the Whitham hierarchy. Such identification, being a particular case of the fundamental formula relating the generating functions or effective actions of quantum theories to the  $\tau$ -functions of the hierarchies of integrable equations, has a little bit implicit form. In this section, I discuss an explicit system of differential equations satisfied by the prepotential  $\mathcal{F}$ . These are the associativity equations [16, 17, 18] and their existence for the prepotential follows, in principle, from the fact that it is  $\tau$ -function of the Whitham hierarchy (though the associativity equations exist even in more general situation [37, 48]). More exactly, in this section I will formulate the most general form of the associativity equations and check their existence in  $2d$  topological models and in the effective Seiberg-Witten theories. Thus, we will deal with the explicit differential equations having among their solutions the prepotentials  $\mathcal{F}$ .

#### 3.1 The associativity equations

The prepotential  $\mathcal{F}$  [20, 21] is defined in terms of a family of Riemann surfaces, endowed with the meromorphic differential  $dS$ . For the gauge group  $G = SU(N)$  the family is [20, 21, 23, 1] given by (36) and the generating differential by (47). The prepotential  $\mathcal{F}(a_i)$  is implicitly defined by the set of equations (119). According to [1], this definition identifies  $\mathcal{F}(a_i)$  as logarithm of (truncated)  $\tau$ -function of Whitham integrable hierarchy. Existing experience with Whitham hierarchies [36, 37] implies that  $\mathcal{F}(a_i)$  should satisfy some sort of the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations [35]. Below in this section we demonstrate that WDVV equations for the prepotential actually look like [16]

$$\mathcal{F}_i \mathcal{F}_k^{-1} \mathcal{F}_j = \mathcal{F}_j \mathcal{F}_k^{-1} \mathcal{F}_i \quad \forall i, j, k = 1, \dots, N-1. \quad (129)$$

Here  $\mathcal{F}_i$  denotes the matrix

$$(\mathcal{F}_i)_{mn} = \frac{\partial^3 \mathcal{F}}{\partial a_i \partial a_m \partial a_n}. \quad (130)$$

Let us first present few comments about equations (129):

- Let us, first, notice that the conventional WDVV equations for topological ( $2d$ ) field theory express the associativity of the algebra  $\phi_i \phi_j = C_{ij}^k \phi_k$  (for symmetric in  $i$  and  $j$  structure constants):  $(\phi_i \phi_j) \phi_k = \phi_i (\phi_j \phi_k)$ , or  $C_i C_j = C_j C_i$ , for the matrices  $(C_i)_n^m \equiv C_{in}^m$ . These conditions become highly non-trivial since, in topological theory, the structure constants are expressed in terms of a single prepotential  $\mathcal{F}(t_i)$ :  $C_{ij}^l = (\eta_{(0)}^{-1})^{kl} \mathcal{F}_{ijk}$ , and  $\mathcal{F}_{ijk} = \frac{\partial^3 \mathcal{F}}{\partial t_i \partial t_j \partial t_k}$ , while the metric is  $\eta_{kl}^{(0)} = \mathcal{F}_{0kl}$ , where  $\phi_0 = I$  is the unity operator. In other words, the conventional WDVV equations can be written as

$$\mathcal{F}_i \mathcal{F}_0^{-1} \mathcal{F}_j = \mathcal{F}_j \mathcal{F}_0^{-1} \mathcal{F}_i. \quad (131)$$

In contrast to (129), in the standard WDVV equations to  $k = 0$ , associated with the distinguished unity operator.

On the other hand, in the Seiberg-Witten effective theory there is no distinguished index  $i$ : all the arguments  $a_i$  of the prepotential can be treated on equal footing. Thus, if some kind of the WDVV

equations holds in this case, it should be invariant under any permutation of indices  $i, j, k$  – criterium satisfied by the system (129). Moreover, the same set of equations (129) is satisfied for generic topological theory.

- In the general theory of Whitham hierarchies [36, 37] the WDVV equations arise also in the form (131). Again, there exists a distinguished time-variable for the global solutions usually associated with the first time-variable of the original KP/KdV hierarchy. Moreover, usually – in contrast to the simplest topological models – the set of these variables for the Whitham hierarchy is infinitely large. In this context our eqs.(129) state that, for specific subhierarchies (in the Seiberg-Witten gluodynamics, it is the Toda-chain hierarchy, associated with a peculiar set of hyperelliptic surfaces), there exists a non-trivial *truncation* of the quasiclassical  $\tau$ -function, when it depends on the finite number ( $N - 1 = g$  – genus of the Riemann surface) of *equivalent* arguments  $a_i$ , and satisfies a much wider set of WDVV-like equations: the whole set (129).
- From (119) it is clear that  $a_i$ ’s are defined modulo linear transformations (one can change  $A$ -cycle for any linear combination of them). Eqs.(129) possess adequate “covariance”: the least trivial part is that  $\mathcal{F}_k$  can be substituted by  $\mathcal{F}_k + \sum_l \epsilon_l \mathcal{F}_l$ . Then

$$\mathcal{F}_k^{-1} \rightarrow (\mathcal{F}_k + \sum_l \epsilon_l \mathcal{F}_l)^{-1} = \mathcal{F}_k^{-1} - \sum_l \epsilon_l \mathcal{F}_k^{-1} \mathcal{F}_l \mathcal{F}_k^{-1} + \sum_l \epsilon_l \epsilon_{l'} \mathcal{F}_k^{-1} \mathcal{F}_l \mathcal{F}_k^{-1} \mathcal{F}_{l'} \mathcal{F}_k^{-1} + \dots \quad (132)$$

Clearly, (129) – valid for all triples of indices *simultaneously* – is enough to guarantee that  $\mathcal{F}_i(\mathcal{F}_k + \sum_l \epsilon_l \mathcal{F}_l)^{-1} \mathcal{F}_j = \mathcal{F}_j(\mathcal{F}_k + \sum_l \epsilon_l \mathcal{F}_l)^{-1} \mathcal{F}_i$ . Covariance under any replacement of  $A$  and  $B$ -cycles together will be seen from the general proof below: in fact the role of  $\mathcal{F}_k$  can be played by  $\mathcal{F}_{d\omega}$ , associated with *any* holomorphic 1-differential  $d\omega$  on the Riemann surface.

- The metric  $\eta$ , which is a second derivative, (as is the case for our  $\eta_{mn}^{(k)} \equiv (\mathcal{F}_k)_{mn}$ ) in ordinary topological theories ( $\eta^{(0)}$ ) is always flat, and this allows one to choose “flat coordinates” where  $\eta^{(0)} = \text{const}$ . Sometimes *all* the metrics  $\eta^{(k)}$  are flat simultaneously. However, this is not always the case: in the example of quantum cohomologies of  $CP^2$  [49, 48] eqs.(129) are true for all  $k = 0, 1, 2$ , but only  $\eta^{(0)}$  is flat while  $\eta^{(1)}$  and  $\eta^{(2)}$  lead to non-vanishing curvatures.

The equations (129) are *trivially* satisfied in the case of  $N = 2$  and  $N = 3$  and become a nontrivial condition only starting with  $N \geq 4$ .

- Our consideration suggests that when the *ordinary* WDVV (131) is true, the whole system (129) holds automatically for any other  $k$  (with the only restriction that  $\mathcal{F}_k$  is non-degenerate). Indeed, <sup>10</sup>

$$\mathcal{F}_i \mathcal{F}_k^{-1} \mathcal{F}_j = \mathcal{F}_0 \left( C_i^{(0)} (C_k^{(0)})^{-1} C_j^{(0)} \right) \quad (133)$$

is obviously symmetric w.r.to the permutation  $i \leftrightarrow j$  implied by  $[C_i^{(0)}, C_j^{(0)}] = 0$ .

- Effective theory (119) is naively *non-topological*. From the 4-dimensional point of view it describes the low-energy limit of the Yang-Mills theory which – at least, in the  $\mathcal{N} = 2$  supersymmetric case – is *not* topological and contains propagating massless particles. Still this theory is entirely defined by a

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<sup>10</sup>This simple proof was suggested by A.Rosly.

prepotential, which – as we now see – possesses *all* essential properties of the prepotentials in topological theory. Thus, from the “stringy” point of view (when everything is described in terms of universality classes of effective actions) the Seiberg-Witten models belong to the same class as topological models: only the way to extract physically meaningful correlators from the prepotential is different. This can serve as a new evidence that the notion of topological theories is deeper than it is usually assumed: as emphasized in [1] it can be actually more related to the low-energy (IR) limit of field theories than to the property of the correlation functions to be constants in physical space-time.

Moreover, the fact that only *third* derivatives enter the equation (129) demonstrates the stringy origin of the nonperturbative solutions *a la* Seiberg-Witten.

### 3.2 The proof of the associativity equations

Let us start with reminding the proof of the WDVV equations (131) for ordinary topological theories. We take the simplest of all possible examples, when  $\phi_i$  are polynomials of a single variable  $\lambda$ . The proof is essentially the check of consistency between the following formulas:

$$\phi_i(\lambda)\phi_j(\lambda) = C_{ij}^k \phi_k(\lambda) \bmod W'(\lambda), \quad (134)$$

$$\mathcal{F}_{ijk} = \text{res} \frac{\phi_i \phi_j \phi_k(\lambda)}{W'(\lambda)} \equiv \sum_{\alpha} \frac{\phi_i \phi_j \phi_k(\lambda_{\alpha})}{W''(\lambda_{\alpha})}, \quad (135)$$

$$\eta_{kl} = \text{res} \frac{\phi_k \phi_l(\lambda)}{W'(\lambda)} \wp \sum_{\alpha} \frac{\phi_k \phi_l(\lambda_{\alpha})}{W''(\lambda_{\alpha})}, \quad (136)$$

$$\mathcal{F}_{ijk} = \eta_{kl} C_{ij}^l. \quad (137)$$

Here  $\lambda_{\alpha}$  are the roots of  $W'(\lambda)$ .

In addition to the consistency of (134)-(137), one should know that *such*  $\mathcal{F}_{ijk}$ , given by (135), are the third derivatives of a single function  $\mathcal{F}(a)$ , i.e.

$$\mathcal{F}_{ijk} = \frac{\partial^3 \mathcal{F}}{\partial a_i \partial a_j \partial a_k}. \quad (138)$$

This integrability property of (135) follows from separate arguments and can be checked independently. But if (134)-(136) is given, the proof of (137) is straightforward:

$$\begin{aligned} \eta_{kl} C_{ij}^l &= \sum_{\alpha} \frac{\phi_k \phi_l(\lambda_{\alpha})}{W''(\lambda_{\alpha})} C_{ij}^l \stackrel{(134)}{=} \\ &= \sum_{\alpha} \frac{\phi_k(\lambda_{\alpha})}{W''(\lambda_{\alpha})} \phi_i(\lambda_{\alpha}) \phi_j(\lambda_{\alpha}) = \mathcal{F}_{ijk}. \end{aligned} \quad (139)$$

Note that (134) is defined modulo  $W'(\lambda)$ , but  $W'(\lambda_{\alpha}) = 0$  at all the points  $\lambda_{\alpha}$ . Imagine now that we change the definition of the metric:

$$\eta_{kl} \rightarrow \eta_{kl}(Q) = \sum_{\alpha} \frac{\phi_k \phi_l(\lambda_{\alpha})}{W''(\lambda_{\alpha})} Q(\lambda_{\alpha}). \quad (140)$$

Then the WDVV equations would still be correct, provided the definition (134) of the algebra is also changed for

$$\phi_i(\lambda)\phi_j(\lambda) = C_{ij}^k(Q)\phi_k(\lambda)Q(\lambda) \bmod W'(\lambda). \quad (141)$$

This describes an associative algebra, whenever the polynomials  $Q(\lambda)$  and  $W'(\lambda)$  are co-prime, i.e. do not have common divisors. Note that (135) – and thus the fact that  $\mathcal{F}_{ijk}$  is the third derivative of the same  $\mathcal{F}$  – remains intact! One can now take for  $Q(\lambda)$  any of the operators  $\phi_k(\lambda)$ , thus reproducing eqs.(129) for all topological theories <sup>11</sup>.

In the case of the Seiberg-Witten model the polynomials  $\phi_i(\lambda)$  are substituted by the canonical holomorphic differentials  $d\omega_i(\lambda)$  on hyperelliptic surface (36). This surface can be represented in a standard hyperelliptic form (37). Instead of (134) and (141) we now put

$$d\omega_i(\lambda)d\omega_j(\lambda) = C_{ij}^k(d\omega)d\omega_k(\lambda)d\omega(\lambda) \bmod \frac{dP_N(\lambda)d\lambda}{y^2}. \quad (142)$$

In contrast to (141) we can not now choose  $Q = 1$  (to reproduce (134)), because now we need it to be a 1-differential. Instead we just take  $d\omega$  to be a *holomorphic* 1-differential. However, there is no distinguished one – just a  $g$ -parametric family – and  $d\omega$  can be *any* one from this family. We require only that it is co-prime with  $\frac{dP_N(\lambda)}{y}$ .

If the algebra (142) exists, the structure constants  $C_{ij}^k(d\omega)$  satisfy the associativity condition (if  $d\omega$  and  $\frac{dP_N}{y}$  are co-prime). But we still need to show that it indeed exists, i.e. that if  $d\omega$  is given, one can find ( $\lambda$ -independent)  $C_{ij}^k$ . This is a simple exercise: all  $d\omega_i$  are linear combinations of

$$dv_k(\lambda) = \frac{\lambda^{k-1}d\lambda}{y}, \quad k = 1, \dots, g: \quad (143)$$

$$dv_k(\lambda) = \sigma_{ki}d\omega_i(\lambda), \quad d\omega_i = (\sigma^{-1})_{ik}dv_k, \quad \sigma_{ki} = \oint_{A_i} dv_k,$$

also  $d\omega(\lambda) = s_k dv_k(\lambda)$ . Thus, (142) is in fact a relation between the polynomials:

$$\left(\sigma_{ii'}^{-1}\lambda^{i'-1}\right)\left(\sigma_{jj'}^{-1}\lambda^{j'-1}\right) = C_{ij}^k\left(\sigma_{kk'}^{-1}\lambda^{k'-1}\right)\left(s_l\lambda^{l-1}\right) + p_{ij}(\lambda)P'_N(\lambda). \quad (144)$$

At the l.h.s. we have a polynomial of degree  $2(g-1)$ . Since  $P'_N(\lambda)$  is a polynomial of degree  $N-1 = g$ , this implies that  $p_{ij}(\lambda)$  should be a polynomial of degree  $2(g-1) - g = g-2$ . The identification of two polynomials of degree  $2(g-1)$  impose a set of  $2g-1$  equations for the coefficients. We have a freedom to adjust  $C_{ij}^k$  and  $p_{ij}(\lambda)$  (with  $i, j$  fixed), i.e.  $g + (g-1) = 2g-1$  free parameters: exactly what is necessary. The linear system of equations is non-degenerate for co-prime  $d\omega$  and  $dP_N/y$ .

Thus, we proved that the algebra (142) exists (for a given  $d\omega$ ) – and thus  $C_{ij}^k(d\omega)$  satisfy the associativity condition

$$C_i(d\omega)C_j(d\omega) = C_j(d\omega)C_i(d\omega). \quad (145)$$

Hence, instead of (135) we have [36, 37, 48]:

$$\begin{aligned} \mathcal{F}_{ijk} &= \frac{\partial^3 \mathcal{F}}{\partial a_i \partial a_j \partial a_k} = \frac{\partial T_{ij}}{\partial a_k} = \\ &= \operatorname{res}_{d\lambda=0} \frac{d\omega_i d\omega_j d\omega_k}{d\lambda \left(\frac{dw}{w}\right)} = \operatorname{res}_{d\lambda=0} \frac{d\omega_i d\omega_j d\omega_k}{d\lambda \frac{dP_N}{y}} = \sum_{\alpha} \frac{\hat{\omega}_i(\lambda_{\alpha})\hat{\omega}_j(\lambda_{\alpha})\hat{\omega}_k(\lambda_{\alpha})}{P'_N(\lambda_{\alpha})/\hat{y}(\lambda_{\alpha})} \end{aligned} \quad (146)$$

The sum at the r.h.s. goes over all the  $2g+2$  ramification points  $\lambda_{\alpha}$  of the hyperelliptic curve (i.e. over the zeroes of  $y^2 = P_N^2(\lambda) - 1 = \prod_{\alpha=1}^N (\lambda - \lambda_{\alpha})$ );  $d\omega_i(\lambda) = (\hat{\omega}_i(\lambda_{\alpha}) + O(\lambda - \lambda_{\alpha}))d\lambda$ ,  $\hat{y}^2(\lambda_{\alpha}) = \prod_{\beta \neq \alpha} (\lambda_{\alpha} - \lambda_{\beta})$ . Formula (146) can be extracted from [36], and its proof is presented, for example, in [16].

<sup>11</sup>To make (129) sensible, one should require that  $W'(\lambda)$  has only *simple* zeroes, otherwise some of the matrices  $\mathcal{F}_k$  can be degenerate and non-invertible.



We define the metric in the following way:

$$\begin{aligned}\eta_{kl}(d\omega) &= \operatorname{res}_{d\lambda=0} \frac{d\omega_k d\omega_l d\omega}{d\lambda \left(\frac{dw}{w}\right)} = \operatorname{res}_{d\lambda=0} \frac{d\omega_k d\omega_l d\omega_k}{d\lambda \frac{dP_N}{y}} = \\ &= \sum_{\alpha} \frac{\hat{\omega}_k(\lambda_{\alpha}) \hat{\omega}_l(\lambda_{\alpha}) \hat{Q}(\lambda_{\alpha})}{P'_N(\lambda_{\alpha})/\hat{y}(\lambda_{\alpha})}\end{aligned}\tag{147}$$

In particular, for  $d\omega = d\omega_k$ ,  $\eta_{ij}(d\omega_k) = \mathcal{F}_{ijk}$ : this choice will give rise to (129).

Given (142), (146) and (147), one can check:

$$\mathcal{F}_{ijk} = \eta_{kl}(d\omega) C_{ij}^k(d\omega).\tag{148}$$

Note that  $\mathcal{F}_{ijk} = \frac{\partial^3 \mathcal{F}}{\partial a_i \partial a_j \partial a_k}$  at the l.h.s. of (148) is independent of  $d\omega$ ! The r.h.s. of (148) is equal to:

$$\begin{aligned}\eta_{kl}(d\omega) C_{ij}^k(d\omega) &= \operatorname{res}_{d\lambda=0} \frac{d\omega_k d\omega_l d\omega}{d\lambda \left(\frac{dw}{w}\right)} C_{ij}^l(d\omega) \stackrel{(142)}{=} \\ &= \operatorname{res}_{d\lambda=0} \frac{d\omega_k}{d\lambda \left(\frac{dw}{w}\right)} \left( d\omega_i d\omega_j - p_{ij} \frac{dP_N d\lambda}{y^2} \right) = \mathcal{F}_{ijk} - \operatorname{res}_{d\lambda=0} \frac{d\omega_k}{d\lambda \left(\frac{dP_N}{y}\right)} p_{ij}(\lambda) \frac{dP_N d\lambda}{y^2} = \\ &= \mathcal{F}_{ijk} - \operatorname{res}_{d\lambda=0} \frac{p_{ij}(\lambda) d\omega_k(\lambda)}{y}\end{aligned}\tag{149}$$

It remains to prove that the last item is indeed vanishing for any  $i, j, k$ . This follows from the fact that  $\frac{p_{ij}(\lambda) d\omega_k(\lambda)}{y}$  is singular only at zeroes of  $y$ , it is not singular at  $\lambda = \infty$  because  $p_{ij}(\lambda)$  is a polynomial of low enough degree  $g - 2 < g + 1$ . Thus the sum of its residues at ramification points is thus the sum over *all* the residues and therefore vanishes.

This completes the proof of associativity equations for the pure  $\mathcal{N} = 2$  SUSY Yang-Mills theory or the Toda chain integrable model [16]. Taking  $d\omega = d\omega_k$  (which is obviously co-prime with  $\frac{dP_N}{y}$ ), we obtain (129).

### 3.3 Algebraic construction of WDVV equations

Let us now discuss in detail the algebraic structure underlying the equations (129) [18, 17]. Remind, first, that for any metric

$$G = \sum_m g^{(m)} \mathcal{F}_m\tag{150}$$

used to raise up indices

$$C_j^{(G)} = G^{-1} \mathcal{F}_j,\tag{151}$$

i.e.  $C_{jk}^i = (G^{-1})^{im} \mathcal{F}_{mjk}$ , or  $\mathcal{F}_{ijk} = G_{im} C_{jk}^m$ <sup>12</sup> the WDVV eqs imply that all matrices  $C$  commute:

$$C_i C_j = C_j C_i \quad \forall i, j\tag{152}$$

(and thus can be diagonalized simultaneously). While (129) implies (152), inverse is not true: the WDVV equations are either (129) or the combination of (150), (151) and (152). Let us also remind that the WDVV eqs (152) expresses the associativity of the multiplication of observables  $\phi_i$  (for example in the chiral rings [57] in  $2d$   $N = 2$  superconformal topological models), where

$$\begin{aligned}\phi_i \circ \phi_j &= C_{ij}^k \phi_k, \\ (\phi_i \circ \phi_j) \circ \phi_k &= \phi_i \circ (\phi_j \circ \phi_k),\end{aligned}\tag{153}$$

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<sup>12</sup>From now on we omit the superscript  $(G)$  in  $C^{(G)}$  and assume summation over repeated indices.

while

$$\mathcal{F}_{ijk} = \langle \langle \phi_i \phi_j \phi_k \rangle \rangle \quad (154)$$

are (deformed) 3-point correlation functions on sphere.

The basic example of the algebra (153) is the multiplication of polynomials modulo  $dP$

$$\phi_i(\lambda)\phi_j(\lambda) = C_{ij}^k \phi_k(\lambda) G'(\lambda) \bmod P'(\lambda) \quad (155)$$

Here  $P(\lambda)$  and  $G(\lambda)$  are polynomials of  $\lambda$ , such that their  $\lambda$ -derivatives  $P'(\lambda)$  and  $G'(\lambda)$  are co-prime (do not have common divisors). The algebra (155) is obviously associative as a factor of explicitly associative multiplication algebra of polynomials over its ideal  $P'(\lambda) = 0$ .

The second ingredient of the WDVV eqs is the residue formula [58],

$$\mathcal{F}_{ijk} = \text{res}_{dP=0} \frac{\phi_i(\lambda)\phi_j(\lambda)\phi_k(\lambda)}{P'(\lambda)} d\lambda \quad (156)$$

In accordance with (151),

$$G'(\lambda) = g^{(m)} \phi_m(\lambda) \quad G_{ij} = g^{(m)} \mathcal{F}_{ijm} \quad (157)$$

The last ingredient is the expression of *flat* moduli  $a_i$  in terms of the polynomial  $P(\lambda)$  [36]:

$$a_i = -\frac{N}{i(N-i)} \text{res} \left( P^{\frac{i}{N}} dG \right), \quad N = \text{ord}(P) \quad (158)$$

These formulas (already used above in the proof of the existence of the WDVV equations in the case of pure gluodynamics) have a straightforward generalization to the case of polynomials of several variables,  $\phi_i(\vec{\lambda}) = \phi_i(\lambda_1, \dots, \lambda_n)$ :

$$\phi_i(\vec{\lambda})\phi_j(\vec{\lambda}) = C_{ij}^k \phi_k(\vec{\lambda}) Q(\vec{\lambda}) \bmod \left( \frac{\partial P}{\partial \lambda_1}, \dots, \frac{\partial P}{\partial \lambda_n} \right), \quad (159)$$

and

$$\mathcal{F}_{ijk} = \text{res}_{dP=0} \frac{\phi_i(\vec{\lambda})\phi_j(\vec{\lambda})\phi_k(\vec{\lambda})}{\prod_{\alpha=1}^n \frac{\partial P}{\partial \lambda_\alpha}} d\lambda_1 \dots d\lambda_n \quad (160)$$

The algebra (155) is always associative, since  $dP = \sum_{\alpha=1}^n \frac{\partial P}{\partial \lambda_\alpha} d\lambda_\alpha$  is always an ideal in the space of polynomials. Moreover, one can even take a factor over generic ideal in the space of polynomials,  $p_1(\vec{\lambda}) = \dots = p_n(\vec{\lambda}) = 0$ , where polynomials  $p_\alpha$  need to be co-prime, but do not need to be derivatives of a single  $P(\vec{\lambda})$ . In this subsection we will discuss in detail the algebraic structure underlying the associativity equations which is more or less natural generalization of the (factorized) polynomial ring structure.

The proof of the WDVV equations in the case of pure  $\mathcal{N} = 2$  gluodynamics presented above does not differ too much from the consideration in the beginning of this subsection except for the substitution of polynomials (functions on a Riemann sphere) holomorphic 1-differentials on Riemann surfaces (complex curves). They always form a family of closed algebras, parametrized by a triple of holomorphic differentials  $dG, d\mathcal{W}, d\Lambda$ . However, these algebras are *not* rings (since the product of two 1-differentials is already a 2-differential), thus they do not give rise immediately to associative algebra after factorization over an ideal. Still, associativity is preserved for many important cases – in particular for the hyperelliptic curves.

The algebraic construction proposed in [18] is interesting because it should possess direct generalizations to higher complex dimensions (from holomorphic 1-forms on complex curves to forms on complex manifolds),

what physically means that one can pass from the WDVV equations on the Seiberg-Witten prepotentials to (hypothetical) universal equations for the prepotentials in string models.

Hence, imagine that in some context the following statements are true:

1. The holomorphic <sup>13</sup> 1-differentials on the complex curve  $\Sigma$  of genus  $g$  form a *closed* algebra,

$$\begin{aligned} d\omega_i(\lambda)d\omega_j(\lambda) &= C_{ij}^k d\omega_k(\lambda)dG(\lambda) + D_{ij}^k d\omega_k(\lambda)d\mathcal{W}(\lambda) + E_{ij}^k d\omega_k(\lambda)d\Lambda(\lambda) = \\ &= C_{ij}^k d\omega_k(\lambda)dG(\lambda) \text{ mod } (d\mathcal{W}, d\Lambda), \end{aligned} \quad (161)$$

where  $d\omega_i(\lambda)$ ,  $i = 1, \dots, g$ , form a complete basis in the linear space  $\Omega^1$  (of holomorphic 1-forms),  $dG$ ,  $d\mathcal{W}$  and  $d\Lambda$  are fixed elements of  $\Omega^1$ , e.g.  $dG(\lambda) = \sum_{m=1}^g \eta^{(m)} d\omega_m$ .

2. The factor of this algebra over the “ideal”  $d\mathcal{W} \oplus d\Lambda$  is associative,

$$C_i C_j = C_j C_i \quad \forall i, j \text{ at fixed } dG, d\mathcal{W}, d\Lambda \quad (162)$$

(remind that  $(C_i)_j^k \equiv C_{ij}^k$ ).

3. The residue formula holds,

$$\frac{\partial \mathcal{F}}{\partial a_i \partial_j \partial a_k} = \text{res}_{d\mathcal{W}=0} \frac{d\omega_i d\omega_j d\omega_k}{d\mathcal{W} d\Lambda} = - \text{res}_{d\Lambda=0} \frac{d\omega_i d\omega_j d\omega_k}{d\mathcal{W} d\Lambda} \quad (163)$$

4. There exists a non-degenerate linear combination of matrices  $\mathcal{F}_i$ .

These statements imply the WDVV eqs (129) for the prepotential  $\mathcal{F}(a_i)$ . Indeed, the substitution of (161) into (163) gives

$$\mathcal{F}_{ijk} = C_{ij}^m G_{mk}, \quad (164)$$

where

$$G_{mk} = \text{res}_{d\mathcal{W}=0} \frac{dG d\omega_m d\omega_k}{d\mathcal{W} d\Lambda} = \eta^{(l)} \mathcal{F}_{lmk}, \quad (165)$$

and the terms with  $d\mathcal{W}$  and  $d\Lambda$  in (161) drop out from  $\mathcal{F}_{ijk}$  because they cancel  $d\Lambda$  or  $d\mathcal{W}$  in the denominator in (163). Eq.(164) can be now substituted into (162) to provide WDVV eqs in the form

$$\mathcal{F}_i G^{-1} \mathcal{F}_j = \mathcal{F}_j G^{-1} \mathcal{F}_i, \quad G = \eta^{(m)} \mathcal{F}_m \quad \forall \left\{ \eta^{(m)} \right\} \quad (166)$$

where at least one invertible metric  $G$  exists by requirement (4).

Existence of the multiplication algebra (161) is rather natural feature of complex curves. Indeed, there are  $g$  holomorphic 1-differentials on the complex curve of genus  $g$ . However, their products  $d\omega_i d\omega_j$  are not linearly independent: they belong to the  $3g - 3$ -dimensional space  $\Omega^2$  of the holomorphic quadratic differentials. Given three holomorphic 1-differentials  $dG$ ,  $d\mathcal{W}$ ,  $d\Lambda$ , one can make an identification

$$\Omega^1 \cdot \Omega^1 \in \Omega^2 \cong \Omega^1 \cdot (dG \oplus d\mathcal{W} \oplus d\Lambda) \quad (167)$$

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<sup>13</sup>Since curves with punctures and the corresponding meromorphic differentials can be obtained by degeneration of smooth curves of higher genera we do not make any distinction between punctured and smooth curves below. We remind that the holomorphic 1-differentials can have at most simple poles at the punctures while quadratic differentials can have certain double poles etc.

which in particular basis is exactly (161). For given  $i, j$  there are  $3g$  adjustment parameters  $C_{ij}^k$ ,  $D_{ij}^k$  and  $E_{ij}^k$  at the r.h.s. of (161), with 3 "zero modes" – in the directions  $dGd\mathcal{W}$ ,  $dGd\Lambda$  and  $d\mathcal{W}d\Lambda$  (i.e. one can add  $d\mathcal{W}$  to  $C_{ij}^k d\omega_k$  and simultaneously subtract  $dG$  from  $D_{ij}^k d\omega_k$ ). Thus we get exactly  $3g - 3$  parameters to match the l.h.s. of (161) – this makes decomposition (161) existing and unique.

Thus we found that the existence of the *closed* algebra (161) is a general feature, in particular it does not make any restrictions on the choice of Riemann surfaces. However, this algebra is not a ring: it maps  $(\Omega^1)^{\otimes 2}$  into *another* space:  $\Omega^1 \otimes \Omega^1 \rightarrow \Omega^2 \neq \Omega^1$ . Thus, its factor over the condition  $d\mathcal{W} = d\Lambda = 0$  is not guaranteed to have all properties of the ring. In particular the factor-algebra

$$d\omega_i \circ d\omega_j = C_{ij}^k d\omega_k \quad (168)$$

does not need to be associative, i.e. the matrices  $C$  alone (neglecting  $D$  and  $E$ ) do not necessarily commute.

However, the associativity would follow if the expansion of  $\Omega^3$  (the space of the holomorphic 3-differentials containing the result of triple multiplication  $\Omega^1 \cdot \Omega^1 \cdot \Omega^1$ ),

$$\Omega^3 = \Omega^1 \cdot dG \cdot dG \oplus \Omega^2 \cdot d\mathcal{W} \oplus \Omega^2 \cdot d\Lambda \quad (169)$$

is unique. Then it is obvious that

$$0 = (d\omega_i d\omega_j) d\omega_k - d\omega_i (d\omega_j d\omega_k) = (C_{ij}^l C_{lk}^m - C_{il}^m C_{jk}^l) d\omega_m dG^2 \text{ mod}(d\mathcal{W}, d\Lambda) \quad (170)$$

would imply  $[C_i, C_k] = 0$ . However, the dimension of  $\Omega^3$  is  $5g - 5$ , while the number of adjustment parameters at the r.h.s. of (169) is  $g + 2(3g - 3) = 7g - 6$ , modulo only  $g + 2$  zero modes (lying in  $\Omega^1 \cdot d\mathcal{W}d\Lambda$ ,  $\Omega^1 \cdot d\mathcal{W}dG^2$  and  $\Omega^1 \cdot d\Lambda dG^2$ ). For  $g > 3$  there is no match:  $5g - 5 < 6g - 8$ , the expansion (169) is not unique, and associativity can (and does) <sup>14</sup> break down unless there is some special reason for it to survive.

This special reason can exist if the curve  $\Sigma$  has specific symmetries. The most important example is the set of curves with an involution  $\sigma : \Sigma \rightarrow \Sigma$ ,  $\sigma^2 = 1$ , such that all  $\sigma(d\omega_i) = -d\omega_i$ , while  $\sigma(d\mathcal{W}) = -d\mathcal{W}$ ,  $\sigma(d\Lambda) = +d\Lambda$ . To have  $d\Lambda$  different from all  $d\omega_i$  one should actually take it away from  $\Omega^1$ , e.g. allow it to be meromorphic.

The most well-known particular example of such curves is the family of hyperelliptic curves described by the equation

$$Y^2 = \text{Pol}_{2g+2}(\lambda), \quad (171)$$

and the involution is  $\sigma : (Y, \lambda) \rightarrow (-Y, \lambda)$ . The space of holomorphic differentials is  $\Omega^1 = \text{Span} \left\{ \frac{\lambda^\alpha d\lambda}{Y(\lambda)} \right\}$ ,  $\alpha = 0, \dots, g-1$ . This space is odd under  $\sigma$ ,  $\sigma(\Omega^1) = -\Omega^1$ , and an example of the (meromorphic) 1-differential which is *even* is

$$d\Lambda = \lambda^r d\lambda, \quad (172)$$

$\sigma(d\Lambda) = +d\Lambda$ . We will assume that  $dG$  and  $d\mathcal{W}$  still belong to  $\Omega^1$  and thus are  $\sigma$ -odd. In the case of hyperelliptic curves with punctures,  $\Omega^1$  can include also  $\sigma$ -even holomorphic 1-differentials (like  $\frac{d\lambda}{(\lambda-\alpha_1)(\lambda-\alpha_2)}$  or just  $d\Lambda$ ), in such cases we consider the algebra (161) of the  $\sigma$ -odd holomorphic differentials  $\Omega_-^1$ , and assume that  $d\omega_i$ ,  $dG$  and  $d\mathcal{W}$  belong to  $\Omega_-^1$ , while  $d\Lambda \in \Omega_+^1$ .

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<sup>14</sup>See [17] for an explicit example of *non*-associativity (actually, this happens in the important Calogero model).

The spaces  $\Omega^2$  and  $\Omega^3$  also split into  $\sigma$ -even and  $\sigma$ -odd parts:  $\Omega^2 = \Omega_+^2 \oplus \Omega_-^2$  and  $\Omega^3 = \Omega_+^3 \oplus \Omega_-^3$ . Multiplication algebra maps  $\Omega_-^1$  into  $\Omega_+^2$  and further into  $\Omega_-^3$ , which have dimensions  $2g-1+2n$  and  $3g-2+3n$  respectively. Here  $n$  enumerates the punctures, where holomorphic 1-differentials are allowed to have simple poles, while quadratic and the cubic ones have at most second- and third-order poles respectively. For our purposes we assume that punctures on the hyperelliptic curves enter in pairs: every puncture is accompanied by its  $\sigma$ -image. Parameter  $n$  is the number of these *pairs*, and the dimension of  $\Omega_-^1$  is  $g+n$ .

Obviously, if all the  $d\omega_i$  in (161) are from  $\Omega_-^1$ , then all  $E_{ij}^k = 0$ , i.e. we actually deal with the decomposition

$$\Omega_+^2 = \Omega_-^1 \cdot dG + \Omega_-^1 \cdot d\mathcal{W} \quad (173)$$

Parameter count now gives:  $2g-1+2n = 2(g+n)-1$  where  $-1$  is for the zero mode  $dGd\mathcal{W}$ . Thus, the hyperelliptic reduction of the algebra (161) does exist.

Moreover, it is associative, as follows from consideration of the decomposition

$$\Omega_-^3 = \Omega_-^1 \cdot dG^2 + \Omega_+^2 \cdot d\mathcal{W} \quad (174)$$

Of crucial importance is that now there is no need to include  $d\Lambda$  in this decomposition, since it does not appear at the r.h.s. of the algebra itself. Parameter count is now:  $3g-2+3n = (g+n) + (2g-1+2n) - 1$  (there is the unique zero mode  $d\mathcal{W}dG^2$ ). Thus, we see that this time decomposition (173) is unique, and our algebra is indeed associative.

In fact, one could come to the same conclusions much easier just noting that all elements of  $\Omega_-^1$  are of the form

$$d\omega_i = \frac{\phi_i(\lambda)d\lambda}{YQ(\lambda)}, \quad (175)$$

where all  $\phi_i(\lambda)$  are polynomials and  $Q(\lambda) = \prod_{i=1}^n (\lambda - m_i)$  is some new polynomial, which takes into account the possible singularities at punctures  $(m_i, \pm Y(m_i))$ . Then our algebra is just the one of the polynomials  $\phi_i(\lambda)$  and it is existing and associative just for the reasons discussed before. The reasoning in this section can be easily modified in the case when hyperelliptic curve possesses an extra involution. The families of such curves appear in the Seiberg-Witten context for the groups  $SO(N)$  and  $Sp(N)$ : the extra involution in these cases is  $\rho: \lambda \rightarrow -\lambda$ . Then one considers  $\Omega_{--}^1$  instead of just  $\Omega_-^1$  (see [17] for further details).

Let us return to the most general consideration of the residue formulas. Consider an integrable model with a Lax operator  $\mathcal{L}(w)$ , which is a  $N \times N$  matrix-valued function (see sect.2 where several examples of such models related to the non-perturbative effective gauge models are considered in details: for them  $N \equiv N_c$ ) on a bare spectral curve  $E$ ,  $w \in E$ , which is usually torus or sphere. Then one can introduce a family of complex curves, defined by the spectral equation (cf. with eqs (35), (50))

$$\det(\mathcal{L}(w) - \lambda) = 0 \quad (176)$$

The family is parametrized by the *moduli* that in this context are values of the  $N$  Hamiltonians of the system (since Hamiltonians commute with each other, these are actually  $c$ -numbers). We obtain this family in a peculiar parametrization, which represents the *full* spectral curves  $\Sigma$  as the ramified  $N$ -sheet coverings over the *bare* curve  $E$ ,

$$\mathcal{P}(\lambda; w) = 0, \quad (177)$$

where  $\mathcal{P}$  is a polynomial of degree  $N$  in  $\lambda$ .

Integrable system is defined by a “generating” 1-form  $dS = \Lambda d\mathcal{W}$ , which possesses the property:

$$\frac{\partial dS}{\partial \text{moduli}} \in \Omega^1, \quad (178)$$

i.e. every variation of  $dS$  with the change of moduli is a holomorphic differential on  $\Sigma$  (normally, even if differential is holomorphic, its moduli-derivative is not).

This structure allows one to define the (subfamily of) holomorphic differentials in a rather explicit form. Let  $s_I$  denote some (specific) coordinates on moduli space  $\mathcal{M}$ . Then

$$\frac{\partial dS}{\partial s_I} \cong \frac{\partial \Lambda}{\partial s_I} d\mathcal{W} = -\frac{\partial \mathcal{P}}{\partial s_I} \frac{d\mathcal{W}}{\mathcal{P}'} \equiv dv_I, \quad (179)$$

and  $dv_I$  provide a set (in general not a canonical set) of holomorphic differentials on  $\Sigma$ . The set of  $dv_I$  is not necessarily the same as  $\Omega^1_-$ , it can be either a subspace of  $\Omega^1_-$  or some  $dv_I$  can be linearly dependent. It is a special requirement (standard in the context of integrable theories) that the differentials  $dv_I$ ’s form a complete basis in  $\Omega^1_-$  (or in  $\Omega^1_{--}$ ).

The prepotential  $\mathcal{F}(a_I)$  is defined by standard formulas (22), (23), (29) and (119). The definition implies in general that the cycles  $A_I$  include  $A_i$ ’s going around the punctures. The conjugate contours  $B_i$  ending in the singularities of  $dS$ .

The self-consistency of the definition (119) of  $\mathcal{F}$ , i.e. the symmetricity of the *period matrix*  $\frac{\partial^2 \mathcal{F}}{\partial a_I \partial a_J}$  is guaranteed by the following reasoning. Let us differentiate equations (119) with respect to moduli  $s_K$  and use (179). Then we get:

$$\int_{B_I} dv_K = \sum_J T_{IJ} \oint_{A_J} dv_K. \quad (180)$$

where the second derivative

$$\frac{\partial^2 \mathcal{F}}{\partial a_I \partial a_J} = T_{IJ} \quad (181)$$

is the period matrix of the (punctured) Riemann surface  $\Sigma$ . As any period matrix, it is symmetric

$$\begin{aligned} \sum_{IJ} (T_{IJ} - T_{JI}) \oint_{A_I} dv_K \oint_{A_J} dv_L &= \sum_I \left( \oint_{A_I} dv_K \int_{B_I} dv_L - \int_{B_I} dv_K \oint_{A_I} dv_L \right) = \\ &= \text{res}(v_K dv_L) = 0 \end{aligned} \quad (182)$$

Note also that the holomorphic differentials associated with the *flat* moduli  $a_I$  are the *canonical*  $d\omega_I$  such that  $\oint_{A_I} d\omega_J = \delta_{IJ}$  and  $\oint_{B_I} d\omega_J = T_{IJ}$ .

In order to derive the residue formula one should now consider the moduli derivatives of the period matrix. It is easy to get:

$$\begin{aligned} \sum_{IJ} \frac{\partial T_{IJ}}{\partial s_M} \oint_{A_I} dv_K \oint_{A_J} dv_L &= \sum_I \left( \oint_{A_I} dv_K \int_{B_I} \frac{\partial dv_L}{\partial s_M} - \int_{B_I} dv_K \oint_{A_I} \frac{\partial dv_L}{\partial s_M} \right) = \\ &= \text{res} \left( v_K \frac{\partial dv_L}{\partial s_M} \right) \end{aligned} \quad (183)$$

The r.h.s. is non-vanishing, since differentiation w.r.t. moduli produces new singularities. From (179)

$$\begin{aligned} -\frac{\partial dv_L}{\partial s_M} &= \frac{\partial^2 \mathcal{P}}{\partial s_L \partial s_M} \frac{d\mathcal{W}}{\mathcal{P}'} + \left( \frac{\partial \mathcal{P}}{\partial s_L} \right)' \left( -\frac{\partial \mathcal{P}}{\partial s_M} \right) \frac{d\mathcal{W}}{\mathcal{P}'} - \frac{\partial \mathcal{P}}{\partial s_L} \frac{\partial \mathcal{P}'}{\partial s_M} \frac{d\mathcal{W}}{(\mathcal{P}')^2} + \frac{\partial \mathcal{P}}{\partial s_L} \frac{\partial \mathcal{P}}{\partial s_M} \frac{\mathcal{P}'' d\mathcal{W}}{(\mathcal{P}')^3} = \\ &= \left[ \left( \frac{\partial \mathcal{P} / \partial s_L \partial \mathcal{P} / \partial s_M}{\mathcal{P}'} \right)' + \frac{\partial^2 \mathcal{P}}{\partial s_L \partial s_M} \right] \frac{d\mathcal{W}}{\mathcal{P}'} \end{aligned} \quad (184)$$

and new singularities (second order poles) are at zeroes of  $\mathcal{P}'$  (i.e. at those of  $d\mathcal{W}$ ). Note that the contributions from the singularities of  $\partial\mathcal{P}/\partial s_L$ , if any, are already taken into account in the l.h.s. of (183). Picking up the coefficient at the leading singularity, we obtain:

$$\text{res}_{v_K} \frac{\partial dv_L}{\partial s_M} = - \text{res}_{d\mathcal{W}=0} \frac{\partial\mathcal{P}}{\partial s_K} \frac{\partial\mathcal{P}}{\partial s_L} \frac{\partial\mathcal{P}}{\partial s_M} \frac{d\mathcal{W}^2}{(\mathcal{P}')^3 d\Lambda} = \text{res}_{d\mathcal{W}=0} \frac{dv_K dv_L dv_M}{d\mathcal{W} d\Lambda} \quad (185)$$

The integrals at the l.h.s. of (183) serve to convert the differentials  $dv_I$  into canonical  $d\omega_I$ . The same matrix  $\oint_{A_I} dv_J$  relates the derivative w.r.t. the moduli  $s_I$  and the periods  $a_I$ . Putting all together we obtain (see also [36]):

$$\begin{aligned} \frac{\partial T_{IJ}}{\partial s^K} &= \text{res}_{d\mathcal{W}=0} \frac{d\omega_I d\omega_J dv_K}{d\mathcal{W} d\Lambda} \\ \frac{\partial^3 F}{\partial a^I \partial a^J \partial a^K} &= \frac{\partial T_{IJ}}{\partial a^K} = \text{res}_{d\mathcal{W}=0} \frac{d\omega_I d\omega_J d\omega_K}{d\mathcal{W} d\Lambda} \end{aligned} \quad (186)$$

Note that these formulas essentially depend only on the symplectic structure  $d\mathcal{W} \wedge d\Lambda$ : e.g. if one makes an infinitesimal shift of  $d\mathcal{W}$  by  $d\Lambda$ , then  $(d\mathcal{W} d\Lambda)^{-1}$  is shifted by  $-(d\mathcal{W})^{-2}$ , i.e. the shift does not contain poles at  $d\Lambda = 0$  and thus does not contribute to the residue formula. Let us note finally, that the above considerations and the residue formulas should be applied literally to the systems described by the holomorphic differentials. In case of curves with the marked points (and, correspondingly, the differentials with poles in these points) the presented above formulas needs some more accurate extra definition, which could be found in [18].

### 3.4 Perturbative example

**The explicit example of the solution to the associativity equations in the framework of the effective Seiberg-Witten theory.** This example corresponds to the perturbative part of the Seiberg-Witten prepotential for  $\mathcal{N} = 2$  SUSY gluodynamics which itself satisfies equations (129). Since the perturbative contribution is non-transcendental, the calculation can be performed in explicit form:

$$\begin{aligned} \mathcal{F}_{pert} \equiv \mathcal{F}(a_i) &= \frac{1}{2} \sum_{\substack{m < n \\ m, n=1}}^N (A_m - A_n)^2 \log(A_m - A_n) \Bigg|_{\sum_m A_m=0} = \\ &= \frac{1}{2} \sum_{\substack{i < j \\ i, j=1}}^{N-1} (a_i - a_j)^2 \log(a_i - a_j) + \frac{1}{2} \sum_{i=1}^{N-1} a_i^2 \log a_i \end{aligned} \quad (187)$$

Here we took  $a_i = A_i - A_N$  – one of the many possible choices of independent variables, which differ by linear transformations. According to (132) the system (129) is covariant under such changes.

Formula (187) means that in 4d SUSY YM theory the perturbative contribution to the effective action has the structure

$$\mathcal{F}_{pert} = \frac{1}{4} \sum_{\text{masses}} (\text{mass})^2 \log(\text{mass}) \quad (188)$$

where in (187) all masses are generated by the v.e.v.'s of the Higgs field by spontaneous breaking mechanism. The formula (188) comes from the requirement that the effective charge

$$\delta^2 \mathcal{F}_{pert} \sim \sum_{\text{masses}} \log(\text{mass}) \quad (189)$$

is one-loop and pure logarithmic.

Let us introduce the notation  $a_{ij} = a_i - a_j$ . The matrix

$$\begin{aligned} \{(\mathcal{F}_1)_{mn}\} &= \left\{ \frac{\partial^3 \mathcal{F}}{\partial a_1 \partial a_m \partial a_n} \right\} = \\ &= \begin{pmatrix} \frac{1}{a_1} + \sum_{l \neq 1} \frac{1}{a_{1l}} & -\frac{1}{a_{12}} & -\frac{1}{a_{13}} & -\frac{1}{a_{14}} & \\ -\frac{1}{a_{12}} & +\frac{1}{a_{12}} & 0 & 0 & \\ -\frac{1}{a_{13}} & 0 & +\frac{1}{a_{13}} & 0 & \dots \\ -\frac{1}{a_{14}} & 0 & 0 & +\frac{1}{a_{14}} & \\ & & & & \dots \end{pmatrix} \end{aligned} \quad (190)$$

i.e.,

$$\begin{aligned} \{(\mathcal{F}_i)_{mn}\} &= \frac{\delta_{mn}(1 - \delta_{mi})(1 - \delta_{ni})}{a_{im}} - \frac{\delta_{mi}(1 - \delta_{ni})}{a_{in}} - \frac{\delta_{ni}(1 - \delta_{mi})}{a_{im}} + \\ &+ \left( \frac{1}{a_i} + \sum_{l \neq i} \frac{1}{a_{il}} \right) \delta_{mi} \delta_{ni} \end{aligned} \quad (191)$$

The inverse matrix

$$\{(\mathcal{F}_k^{-1})_{mn}\} = a_k + \delta_{mn} a_{km} (1 - \delta_{mk}), \quad (192)$$

for example

$$\{(\mathcal{F}_1^{-1})_{mn}\} = a_1 \begin{pmatrix} 1 & 1 & 1 & . \\ 1 & 1 & 1 & . \\ 1 & 1 & 1 & . \\ \dots & & & \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & . \\ 0 & a_{12} & 0 & . \\ 0 & 0 & a_{13} & . \\ \dots & & & \end{pmatrix} \quad (193)$$

As the simplest example let us consider the case  $N = 4$ . We already know that for  $N = 4$  it is sufficient to check only one of the eqs.(129), all the others follow automatically. We take  $k = 1$ . Then,

$$\begin{aligned} \mathcal{F}_1 &= \begin{pmatrix} \frac{1}{a_1} + \frac{1}{a_{12}} + \frac{1}{a_{13}} & -\frac{1}{a_{12}} & -\frac{1}{a_{13}} \\ -\frac{1}{a_{12}} & \frac{1}{a_{12}} & 0 \\ -\frac{1}{a_{13}} & 0 & \frac{1}{a_{13}} \end{pmatrix} \quad \mathcal{F}_2^{-1} = \begin{pmatrix} a_2 + a_{21} & a_2 & a_2 \\ a_2 & a_2 & a_2 \\ a_2 & a_2 & a_2 + a_{23} \end{pmatrix} \\ \mathcal{F}_3 &= \begin{pmatrix} \frac{1}{a_{31}} & 0 & -\frac{1}{a_{31}} \\ 0 & \frac{1}{a_{32}} & -\frac{1}{a_{32}} \\ -\frac{1}{a_{31}} & -\frac{1}{a_{32}} & \frac{1}{a_3} + \frac{1}{a_{31}} + \frac{1}{a_{32}} \end{pmatrix} \end{aligned} \quad (194)$$

and, say,

$$\mathcal{F}_1 \mathcal{F}_2^{-1} \mathcal{F}_3 = \begin{pmatrix} \star & -\frac{1}{a_{31}} & \Delta + \frac{a_{21} + a_{23}}{a_{13}^2} \\ -\frac{1}{a_{13}} & \star & \frac{1}{a_{13}} \\ \frac{a_{21} + a_{23}}{a_{13}^2} & \frac{1}{a_{13}} & \star \end{pmatrix} \quad (195)$$



where we do not write down manifestly the diagonal terms since, to check (129), one only needs to prove the symmetricity of the matrix. This is really the case, since

$$\Delta \equiv \frac{a_2}{a_1 a_3} - \frac{a_{21}}{a_1 a_{31}} - \frac{a_{23}}{a_3 a_{13}} = 0 \quad (196)$$

Only at this stage we use manifestly that  $a_{ij} = a_i - a_j$ .

Now let us prove (129) for the general case. We check the equation for the inverse matrices. Namely, using formulas (191)-(192), one obtains

$$\begin{aligned} & (\mathcal{F}_i^{-1} \mathcal{F}_j \mathcal{F}_k^{-1})_{\alpha\beta} = \\ &= \frac{a_i a_k}{a_j} + \delta_{\alpha\beta} (1 - \delta_{i\alpha})(1 - \delta_{k\alpha})(1 - \delta_{j\alpha}) \frac{a_{i\alpha} a_{k\beta}}{a_{j\beta}} + \delta_{j\alpha} \delta_{j\beta} (1 - \delta_{i\alpha})(1 - \delta_{k\beta}) \left( \frac{1}{a_j} + \sum_{n \neq j} \frac{1}{a_{jn}} \right) + \\ &+ \delta_{j\alpha} (1 - \delta_{i\alpha}) a_{i\alpha} \left( \frac{a_k}{a_j} - \frac{a_{k\beta}}{a_{j\beta}} (1 - \delta_{k\beta})(1 - \delta_{j\beta}) \right) + \delta_{j\beta} (1 - \delta_{k\beta}) \left( \frac{a_i}{a_j} - \frac{a_{i\alpha}}{a_{j\alpha}} (1 - \delta_{i\alpha})(1 - \delta_{j\alpha}) \right) = \quad (197) \\ &= \frac{a_i a_k}{a_j} + \delta_{\alpha\beta} (1 - \delta_{i\alpha} - \delta_{k\alpha} - \delta_{j\alpha}) \frac{a_{i\alpha} a_{k\beta}}{a_{j\beta}} + \delta_{j\alpha} \delta_{j\beta} \left( \frac{1}{a_j} + \sum_{n \neq j} \frac{1}{a_{jn}} \right) + \\ &+ \delta_{j\alpha} a_{i\alpha} \left( \frac{a_k}{a_j} - \frac{a_{k\beta}}{a_{j\beta}} (1 - \delta_{k\beta} - \delta_{j\beta}) \right) + \delta_{j\beta} \left( \frac{a_i}{a_j} - \frac{a_{i\alpha}}{a_{j\alpha}} (1 - \delta_{i\alpha} - \delta_{j\alpha}) \right) \end{aligned}$$

where we used that  $i \neq j \neq k$ . The first three terms are evidently symmetric with respect to interchanging  $\alpha \leftrightarrow \beta$ . In order to prove the symmetricity of the last two terms, we need to use the identities  $\frac{a_k}{a_j} - \frac{a_{k\beta}}{a_{j\beta}} = \frac{a_{\beta} a_{jk}}{a_j a_{j\beta}} \xrightarrow{k=\beta} \frac{a_k}{a_j}$ ,  $\frac{a_i}{a_j} - \frac{a_{i\alpha}}{a_{j\alpha}} = \frac{a_{\alpha} a_{ji}}{a_j a_{j\alpha}} \xrightarrow{i=\alpha} \frac{a_i}{a_j}$ . Then, one gets

$$\text{the last line of (197)} = \delta_{j\alpha} (1 - \delta_{j\beta}) \frac{a_{ij} a_{jk}}{a_j} \frac{a_{\beta}}{a_{j\beta}} + \delta_{j\beta} (1 - \delta_{j\alpha}) \frac{a_{ij} a_{jk}}{a_j} \frac{a_{\alpha}}{a_{j\alpha}} + \delta_{j\alpha} \delta_{j\beta} \frac{a_k a_{i\alpha} + a_i a_{k\beta}}{a_j} \quad (198)$$

It is interesting to note that in the particular example (187), all the metrics  $\eta^{(k)}$  are flat. Moreover, it is easy to find the explicit flat coordinates:

$$\begin{aligned} \eta^{(k)} &= \eta_{ij}^{(k)} da^i da^j = \mathcal{F}_{ijk} da^i da^j = da_i da_j \partial_{ij}^2 (\partial_k \mathcal{F}) = \\ &= \frac{da_k^2}{a_k} + \sum_{l \neq k} \frac{da_{kl}^2}{a_{kl}} = 4 \left( (d\sqrt{a_k})^2 + \sum_{l \neq k} (d\sqrt{a_{kl}})^2 \right). \end{aligned} \quad (199)$$

The explicit form of non-perturbative (instantonic etc) contributions to the prepotential are more complicated. For several examples their computation from the exact formulas *a la* Seiberg-Witten can be found in [17] and the discussion of consistency with standard quantum field theory methods can be found in [59].

**Holomorphic differentials on a punctured sphere.** Let us show now that the perturbative example corresponds to *rational* degeneration of the spectral curve, namely to the Riemann sphere with some punctures at the points  $\lambda_i$ ,  $i = 1, \dots, N$ , so that the canonical basis in the space  $\Omega^1$  can be chosen as:

$$d\omega_i = \frac{(\lambda_i - \lambda_N) d\lambda}{(\lambda - \lambda_i)(\lambda - \lambda_N)}, \quad i = 1, \dots, N-1 \quad (200)$$

We assumed that the  $A_i$  cycles wrap around the points  $\lambda_i$ , while their conjugated  $B_i$  connect  $\lambda_i$  with the reference puncture  $\lambda_N$ . Multiplication algebra of  $d\omega_i$ 's is defined modulo

$$d\mathcal{W} = d \log P_N(\lambda) = \frac{dP_N(\lambda)}{P_N(\lambda)}, \quad (201)$$

$P_N(\lambda) = \prod_{i=1}^N (\lambda - \lambda_i)$ , and it is obviously associative.

The periods  $a_i$  depend on the choice of the generating differential  $dS = \Lambda dW$ . There are two essentially different choices  $\Lambda = \lambda$  [1] and  $\Lambda = \log \lambda$  [19], i.e.

$$dS^{(4)} = \lambda d \log P_N(\lambda) \quad \text{and} \quad dS^{(5)} = \log \lambda d \log P_N(\lambda) \quad (202)$$

In order to fulfill the requirement (178) one should assume that  $\sum_{i=1}^N \lambda_i = 0$  in the case of  $dS^{(4)}$ , while  $\prod_{i=1}^N \lambda_i = 1$  in the case of  $dS^{(5)}$ . Since  $A_i$  cycle just wraps around the point  $\lambda = \lambda_i$ , the  $A_i$ -periods of such  $dS$  are

$$\begin{aligned} a_i^{(4)} &= \oint_{\lambda_i} dS^{(4)} = \lambda_i, \\ a_i^{(5)} &= \oint_{\lambda_i} dS^{(5)} = \log \lambda_i \end{aligned} \quad (203)$$

The corresponding residue formulas are

$$\begin{aligned} \mathcal{F}_{ijk}^{(4)} &= \sum_{m=1}^N \text{res}_{\lambda_m} \frac{d\omega_i d\omega_j d\omega_k}{d\lambda d \log P_N}, \\ \mathcal{F}_{ijk}^{(5)} &= \sum_{m=1}^N \text{res}_{\lambda_m} \lambda \frac{d\omega_i d\omega_j d\omega_k}{d\lambda d \log P_N}, \quad i, j, k = 1, \dots, N-1 \end{aligned} \quad (204)$$

and they both provide solutions to the WDVV equations. The prepotentials are (187) and [17, 18]:

$$\begin{aligned} \mathcal{F}^{(5)}(a_i) &= \sum_{1 \leq i < j \leq N} \tilde{Li}_3(e^{a_i - a_j}) - \frac{N}{2} \sum_{1 \leq i < j < k \leq N} a_i a_j a_k, \quad \sum_{i=1}^N a_i = 0, \\ \partial_x^2 \tilde{Li}_3(e^x) &\equiv \log 2 \sinh x, \quad \tilde{Li}_3(e^x) = \frac{1}{6} x^3 - \frac{1}{4} Li_3(e^{-2x}) \end{aligned} \quad (205)$$

and describe the perturbative limit of the  $N = 2$  supersymmetric  $SU(N)$  gauge models in  $4d$  and  $5d$  [19] respectively. Note only that the expression (205) differs from that of [19] on cubic in periods  $\mathbf{a}$  terms, whose presence is *necessary* for the prepotential to satisfy the WDVV equations.

If the punctures  $\lambda_i$  are not all independent, the same formulas provide solutions to the WDVV equations, associated with the other simple groups:  $SO(N)$ ,  $Sp(N)$ ,  $F_4$  and  $E_{6,7,8}$  ( $G_2$  does not have enough moduli to provide non-trivial solutions to the WDVV eqs). If  $P_N$  is substituted by

$$P_N \rightarrow \frac{P_N}{Q_{N_f}^{1/2}} = \frac{\prod_{i=1}^N (\lambda - \lambda_i)}{\prod_{\ell=1}^{N_f} (\lambda - m_\ell)^{1/2}}, \quad (206)$$

one gets solutions, interpreted as (perturbative limits of) the gauge models with matter supermultiplets in the first fundamental representation. Inclusion of matter in other representations seems to destroy the WDVV equations, at least, generically; note that such models do not arise in a natural way from string compactifications, and there are no known curves associated with them in the Seiberg-Witten theory (see [17] for details).

**Holomorphic differentials on hyperelliptic curves.** *Non-perturbative* deformations of the above prepotentials arise when the punctures on Riemann sphere are blown up to form handles of the hyperelliptic curve:

$$\begin{aligned} W + \frac{1}{W} &= 2 \frac{P_N(\lambda)}{Q(\lambda)_{N_f}^{1/2}}, \\ W - \frac{1}{W} &= 2 \frac{Y(\lambda)}{Q(\lambda)_{N_f}^{1/2}}, \end{aligned} \quad (207)$$

$$Y^2(\lambda) = P_N^2(\lambda) - Q_{N_f}(\lambda)$$

These curves, together with the corresponding differentials  $dS$

$$dS^{(4)} = \lambda \frac{dW}{W}, \quad dS^{(5)} = \log \lambda \frac{dW}{W}, \quad (208)$$

(i.e.  $dW = \frac{d\lambda}{\lambda}$  and  $d\Lambda^{(4)} = d\lambda$ ,  $d\Lambda^{(5)} = \frac{d\lambda}{\lambda}$ ) are implied by integrable models of the Toda-chain family [1, 2, 3, 19]. Together with the residue formula (163) these provide the nonperturbative solution to the WDVV equations.

For example the proof of subsect.3.2 can be almost literally transferred to the case of the relativistic Toda chain system corresponding to the  $5d$   $N = 2$  SUSY pure gauge model with one compactified dimension [19]. The main reason is that in the case of relativistic Toda chain the spectral curve is a minor modification of (45) having the form

$$w + \frac{1}{w} = (\zeta\lambda)^{-N_c/2} P(\lambda), \quad (209)$$

which can be again rewritten as a *hyperelliptic* curve in terms of the new variable  $Y \equiv (\zeta\lambda)^{N_c/2} (w - \frac{1}{w})$

$$Y^2 = P^2(\lambda) - 4\zeta^{2N_c} \lambda^{N_c} \quad (210)$$

where  $\lambda \equiv e^{2\xi}$ ,  $\xi$  is the "true" spectral parameter of the relativistic Toda chain and  $\zeta$  is its coupling constant.

The difference with the  $4d$  case (see subsect.3.2) is the following two points. The first is that now  $s_0 \sim \prod e^{a_i} = 1$  while  $s_{N_c-1}$  becomes an unfrozen moduli parameter.

The second essential new point is that the generating differential instead of (47) in five-dimensional case is

$$dS = \xi \frac{dw}{w} \sim \log \lambda \frac{dw}{w} \quad (211)$$

so that

$$dW_k = \frac{\partial dS}{\partial s_k} \cong \frac{\lambda^{k-1} d\lambda}{Y}, \quad k = 1, \dots, g \quad (212)$$

Despite the condition  $s_0 = 1$ , i.e. absence of the corresponding module, this formula literally coincides (because of the additional degree  $\lambda$  in the denominator) with formula (143). Thus, the algebra of differentials remains the same associative algebra, the only difference being slightly modified residue formula because of modifying the polynomial  $P$  and, therefore, the differential  $d\omega$ . Namely, the residue formula acquires the following form

$$F_{ijk} = \text{res}_{d\lambda=0} \frac{d\omega_i d\omega_j d\omega_k}{\left(\frac{d\lambda}{\lambda}\right) \left(\frac{dw}{w}\right)} = \text{res}_{d\lambda=0} \frac{d\omega_i d\omega_j d\omega_k}{\left(\frac{d\lambda}{\lambda}\right) \left(\frac{dP}{Y}\right)} = \sum_{\alpha} \lambda_{\alpha} \frac{\hat{\omega}_i(\lambda_{\alpha}) \hat{\omega}_j(\lambda_{\alpha}) \hat{\omega}_k(\lambda_{\alpha})}{P'(\lambda_{\alpha}) / \hat{Y}(\lambda_{\alpha})} \quad (213)$$

**Other examples.** The very natural question is what happens with the WDVV equations for Toda chain models, associated with the exceptional groups. The problem is that the associated spectral curves are not hyperelliptic – at least naively. Still they have enough symmetries to make our general reasoning working, but this requires a special investigation.

The number of examples can be essentially increased by the study of various integrable hierarchies, peculiar configurations of punctures etc. In recent paper [60] it was actually suggested that – at least in peculiar models –  $dS$  can be expressed through the Baker-Akhiezer function:  $dS = \Lambda d \log \Psi$ . Of more importance could be crucially interesting lift to  $6d$  or elliptic models (in the same sense as five-dimensional theory is a cylindric or trigonometric  $\lambda \rightarrow \log \lambda$  generalization of the four-dimensional theory) which requires interpretation of  $\lambda$  as a coordinate on elliptic curve. It is not yet known, if this transition breaks down the WDVV equations.

## 4 Conclusion

In these notes I have tried to present the main ingredients of the theory of integrable systems which appeared recently to be rather useful to understand the nonperturbative results in quantum strings and supersymmetric gauge theories. The most exciting thing in this picture is that there exists an effective description (by means of classical and finite-dimensional integrable models) of the theory which is quantum (infinite-dimensional!) field theory, contains propagating (massless) particles and is *not* a quantum integrable model at least in conventional sense.

Hypothetically, a generalization to realistic string models is straightforward and related first of all with the prepotentials arising in the study of the models related to the Calabi-Yau compactifications. The steps described above can be in principle repeated leading to the integrable models based on the *higher-dimensional complex* manifolds (instead of 1D-dimensional *curves*  $\Sigma$ ). Such integrable systems are not investigated yet in detail (see however [38, 50]) and are supposed to be much more complicated than the well-known integrable systems of KP/Toda type.

One more direction (which was almost not considered in the text above) is related to the study of effective theories on (partially) compactified target-spaces [51, 19]. Adding one compactified dimension leads to appearance of the well-known class of relativistic integrable models [52] and allows one to interpret the divisor on a complex curve corresponding to a finite-gap solution in terms of the nonlocal observables (loops). Thus, on one hand it should clarify the sense of arising integrable systems, while on the other hand it is a step towards study of the prepotentials of string models (see for example [61]) where there are contributions having similar (though more complicated) structure.

In spite of all the problems it is easy to believe that for all the theories where it is possible to make any statement about the nonperturbative and exact quantities there exists something more than a summation of a perturbation theory. The main idea advocated above and had been checked already in several examples is based on general belief that the realistic theory should be selfconsistent and adjust automatically its properties not to be ill-defined both at large and small distances. It looks that an adequate nonperturbative language for the effective formulation of consistent in this sense field and string theories can be looked for among integrable systems.

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